Draft

Basics of Structural Reliability

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Abstract. A basic summary is given on the theory of probability, statistical assessments and the aspects of inference and model building required as a basis for understanding the fundamentals of structural reliability. Basic concepts are outlined concerning the aspects of probabilistic modeling of load and resistance variables including the aspects of load combinations and materials models established of basic physical characteristics. Finally the aspects of reliability of technical components is addressed and First Order Reliability Methods are introduced.

Introduction

The present lecture notes have been written for the purpose of giving an introduction to the theory of probability and structural reliability and an overview of some of the most important philosophies and aspects of probabilistic modeling and reliability analysis in engineering applications. In no manner should the notes be taken for being neither complete nor comprehensive in the coverage of theories and the implications of these on practical applications.

The notes have been written on the basis of various standard texts including Benjamin and Cornell [1], Melchers [2], Schneider [3], Madsen [4], Thoft-Christensen and Baker [5], DuraCrete [6] as well as my own lecture notes for the course "Risiko und Sicherheit" presently being taught on the 6'th semester at Institut für Baustatik und Konstruktion at the Federal Institute of Technology, ETH, Zürich.

1 Basic Probability Theory

1.1 Introduction

In our daily lives words like risk, chance, likelihood and probability are being used interchangeably to indicate that we are uncertain about the state of the item or issue under discussion. For example we talk about the risk of getting cancer due to cigarette smoking, the chance that we will succeed in developing a vaccine against the HIV virus in 2001, the likelihood of a getting a "Royal Flush" in a Poker game and the probability of getting a major earth-quake in the Bay area within the next decade.
The level of uncertainty may be expressed as purely qualitative statements such as "the chance is good" or "the probability is low" but may also be quantified in terms of numbers or percentages.
In the above examples the different words in fact all have the meaning of probability and in the following section we will investigate this notion and especially the theoretical framework for its quantification in more detail.

1.2 Definition of probability

The purpose of the theory of probability is to enable the quantitative assessment of probabilities but the real meaning and interpretation of probabilities and probabilistic calculations as such is not a part of the theory. Consequently two people may have completely different interpretations of the probability concept, but still use the same calculus.

1.2.1 Mathematical definition

The mathematical definition of probability is based on the following two axioms.

Axiom 1:

The probability of an event A, i.e. P(A) is always smaller than or equal to 1 and larger than or equal to 0, i.e. P(A) is bounded as

$$0 \le P(A) \le 1 \tag{1}$$

where the lower bound 0 corresponds to the situation that that the event A with certainty will not occur and the upper bound 1 to the situation that the event A with certainty will occur. In intermediate cases the event A 'might occur' with increasing probability.

Axiom 2:

Given that the events A and B are mutually exclusive, i.e. cannot occur simultaneously the probability of the event A or the event B is given by

$$P(A \bigcup B) = P(A) + P(B) \tag{2}$$

These two properties together are sufficient to build up the theory of probability. However, before we will defer to this we will show some of the further definitions and statements.

1.2.2 Frequentistic definition

The frequentistic definition of probability is the typical interpretation of probability of the experimentalist. In this interpretation the probability P(A) is simply the relative frequency of occurrence of the event A as observed in an experiment with n trials, i.e. the probability of an event A is defined as the number of times that the event A occurs divided by the number of experiments that is carried out.

$$P(A) = \lim \frac{N_A}{n_{\text{exp}}} \qquad \text{for} \qquad n_{\text{exp}} \to \infty \tag{3}$$

N_A = number of experiments where A occurred

n_{exp} = total number of experiments.

If a frequentist is asked what the probability is for achieving a “head” when flipping a coin he would principally not know what to answer until he would have performed a large number of experiments. If say after 1000 experiments (flips with the coin) it is observed that “head” has occurred 563 times the answer would be that the probability for “head” is 0.563. However, as the number of experiments is increased the probability would converge towards 0.5.

1.2.3 Classical definition

The classical definition refers to the days that the probability calculus was founded by Pascal and Fermat. The inspiration for this theory was found in the games of cards and dices. The classical definition of the probability of the event A can be formulated as:

$$P(A) = \frac{n_A}{n_{tot}} \tag{4}$$

n_A = number of equally likely ways an experiment may lead to A

n_{tot} = total number of equally likely ways in the experiment

According to the classical definition of probability the probability of achieving a “head” when flipping a coin would be 0.5 as there is only one possible way to achieve a “head” and there are two equally likely outcomes of the experiment. In fact there is no real contradiction to the frequentistic definition but the following differences may be observed

- that the experiment need not to be carried out as the answer is known in advance
- that the classical theory gives no solution unless all possible equally possible ways may be derived analytically.

1.2.4 Bayesian definition

In the Bayesian (Bayes) interpretation the probability P(A) of the event A is formulated as a degree of belief that A will occur:

$$P(A) = \text{ degree of belief that } A \text{ will occur} \tag{5}$$

Coming back to the coin-flipping problem the Bayesian would argue that there are two possibilities, and as he has no preferences to "head" or "tail" he would judge the probability of achieving a "head" to be 0.5.

The degree of belief is in reality a reflection of the state of mind of the individual person in terms of experience, expertise and preferences. In this respect the Bayesian interpretation of probability is subjective or more precisely person dependent. This opens up the possibility that two different persons may assign the probability of a given event differently in this sense is in contradiction to the frequentistic interpretation where probabilities are assumed to be a characteristic of nature.
The Bayesian statistical interpretation of probability includes the frequentistic and the classical interpretation in the sense that the subjectively assigned probabilities may be based on the experience from previous experiments (frequentistic) as well as considerations of e.g. symmetry (classical).
As we will discuss further in a later section the degree of belief is also referred to as a prior belief or prior probability, i.e. the belief, which may be assigned prior to obtaining any further knowledge. It is interesting to note that Kant developed the pure philosophical basis for the treatment of subjectivity at the same time as Bayes developed the mathematical framework later known as the Bayesian statistics.
The modern structural reliability is based on the Bayesian interpretation of probability. However, the degree of freedom in the assignment of probabilities is in reality not so large as indicated in the above. In a formal Bayesian framework, the subjective element should be formulated before the relevant data are observed. Arguments of objective symmetrical reasoning and physical constraints, of course, should be taken into account.

1.2.5 Summary

In some cases probabilities may adequately be assessed by means of frequentistic information. This is e.g. the case when the probability of failure of mass produced components are considered, such as pumps, light bulbs and valves. However, in order to utilize reported failures for the assessment of probability of failure for such components it is a prerequisite that the components are in principle identical, that they have been subject to the same operational and/or loading conditions and that the failures can be assumed to be independent.
In other cases when the considered components are e.g. bridges, high-rise buildings, ship structures or unique configurations of pipelines and pressure vessels these conditions are not fulfilled. First of all the number of in principle identical structures may be very small (or even just one) and the conditions in terms of operational and loading conditions are normally significantly different from structure to structure. In such cases the Bayesian statistics is far more appropriate.
The basic idea behind the Bayesian statistics being that lack of knowledge is an uncertainty that should be treated by probabilistic reasoning in a similar way to other types of uncertainty. In reality decisions have to be made despite the lack of knowledge and statistical tools are a great help in that process.

1.3 Sample space and events

Considering e.g. the compression strength of concrete this characteristic may be determined by performing laboratory experiments on standardized test specimens (cylinders or cubes). Each test result will, however, likely be different from the other and the concrete compression strength must be assumed to be an uncertain quantity or a random quantity. The set of all possible outcomes of the concrete compression strength experiments is called the sample space Ω for the random quantity – the concrete compression strength. In this example the sample space is the open interval $\Omega =]0;\infty[$, i.e. the set of all positive real numbers. In this case the sample space is furthermore continuous but in other cases e.g. when considering the outcome of throwing dices the sample space can also be discrete and countable.

An event is defined as a subset of a sample space and thus a set of sample points. If the subset is empty, i.e. contains no sample points it is said to be impossible. An event is said to be certain if it contains all sample points in the sample space, i.e. the event is identical to the sample space.

Consider the two events E_1 and E_2 shown in Figure 1. The subset of sample points belonging to the event E_1 and/or the event E_2 is denoted the union of the events E_1 and E_2 written as $E_1 \bigcup E_2$.

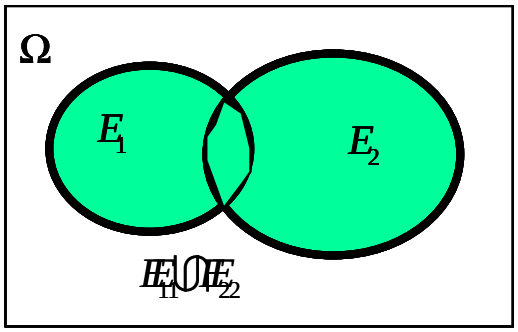


Figure 1 Venn diagrams illustrating the union of events (right) and the intersection of events (left)

The subset of sample points belonging to E_1 and E_2 is denoted the intersection of E_1 and E_2 and is written as $E_1 \bigcap E_2$.

The two events are said to be mutually exclusive if they are disjoint, i.e. they have no common sample points. In this case the intersection of E_1 and E_2 is empty, i.e. $E_1 \bigcap E_2 = \varnothing$ where $\varnothing$ is the empty set.

Consider the event E in the sample space Ω. The event containing all sample points in Ω, which are not included in E are called the complementary event to E and denoted $\overline{E}$. It then follows directly that $E \bigcup \overline{E} = \Omega$ and that $E \bigcap \overline{E} = \varnothing$.

It can be shown that the intersection and union operations obey the following commutative, associative and distributive laws

$$
\begin{aligned}
&E_1 \bigcap E_2 = E_2 \bigcap E_1 \text{ and } E_1 \bigcup E_2 = E_2 \bigcup E_1 \\
&E_1 \bigcap (E_2 \bigcap E_3) = (E_1 \bigcap E_2) \bigcap E_3 \\
&E_1 \bigcup (E_2 \bigcup E_3) = (E_1 \bigcup E_2) \bigcup E_3 \\
&E_1 \bigcap (E_2 \bigcup E_3) = (E_1 \bigcap E_2) \bigcup (E_1 \bigcap E_3) \\
&E_1 \bigcup (E_2 \bigcap E_3) = (E_1 \bigcup E_2) \bigcap (E_1 \bigcup E_3)
\end{aligned}
\tag{6}
$$

From which the so-called De Morgan's laws may be derived

$$
\begin{aligned}
&E_1 \bigcap E_2 = \overline{\overline{E}_1 \bigcup \overline{E}_2} \\
&E_1 \bigcup E_2 = \overline{\overline{E}_1 \bigcap \overline{E}_2}
\end{aligned}
\tag{7}
$$

1.4 The three axioms of probability theory

As already indicated the probability theory is mathematically build up be the following (only) three axioms

Axiom 1

$$0 \le P(E) \le 1 \tag{8}$$

where P is the probability measure.

Axiom 2

$$P(\Omega) = 1 \tag{9}$$

where Ω is the sample space.

Axiom 3

Given that $E_1, E_2, ..E_n$ are mutually exclusive events then

$$P(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} P(E_i) \tag{10}$$

These three axioms is the sole basis for the theory of probability.

1.5 Conditional probability and Bayes rule

Conditional probabilities are of special interest in risk and reliability analysis as these for the basis for updating the prior probability estimates based of new information, knowledge and evidence.

The conditional probability of the event E_1 given that the event E_2 has occurred is written as

$$P(E_1 | E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)} \tag{11}$$

It is seen that the conditional probability is not defined if the conditioning event is the empty set, i.e. when $P(E_2) = 0$.

The event E_1 is said to be statistically independent of the event E_2 if

$$P(E_1 | E_2) = P(E_1) \tag{12}$$

implying that the occurrence of the event E_2 does not affect the probability of E_1.

From Equation (11) the probability of the event $E_1 \cap E_2$ may be given as

$$P(E_1 \cap E_2) = P(E_1 | E_2) \cdot P(E_2) \tag{13}$$

and it follows immediately that if the events E_1 and E_2 are independent that

$$P(E_1 \cap E_2) = P(E_1) \cdot P(E_2)$$

Based on the above findings the important rule of Bayes may now be derived.

Consider the sample space Ω divided into n mutually exclusive events $E_1, E_2, ..E_n$ (see also Figure 2 where the case of n = 8 is considered).

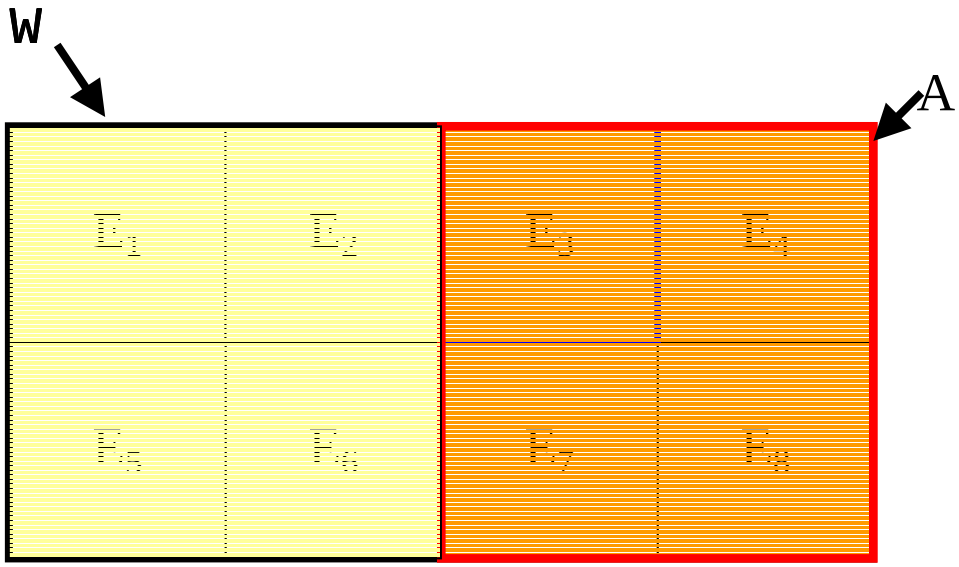


Figure 2 Illustration of the rule of Bayes.

Furthermore let the event A be an event in the sample space Ω. Then we can write the probability of the event A, i.e. P(A) as

$$\begin{aligned} &P(A) = P(A \cap E_1) + P(A \cap E_2) + \ldots + P(A \cap E_n) \\ &P(A|E_1) \cdot P(E_1) + P(A|E_2) \cdot P(E_2) + \ldots + P(A|E_n) \cdot P(E_n) = \\ &\sum_{i=1}^{n} P(A|E_i) \cdot P(E_i) \end{aligned} \tag{14}$$

this is also referred to as the total probability theorem.

From Equation (11) we have that

$$P(A|E_i) \cdot P(E_i) = P(E_i|A) \cdot P(A) \qquad \text{implying that}$$

$$P(E_i|A) = \frac{P(A|E_i) \cdot P(E_i)}{P(A)} \tag{15}$$

Now by inserting Equation (14) into Equation (15) we get the rule due to Bayes

$$P(E_i|A) = \frac{P(A|E_i) \cdot P(E_i)}{\sum_{j=1}^{n} P(A|E_j) \cdot P(E_j)} \tag{16}$$

In Equation (16) the conditional term $P(A|E_i)$ are often referred to as the likelihood, i.e. the probability of observing a certain state given the true state. The term $P(E_i)$ is the prior probability of the event E_i, i.e. prior to the knowledge about the event A.

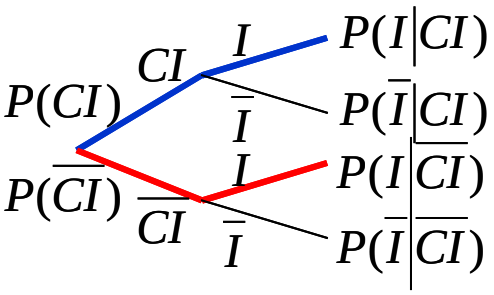
As mentioned previously the rule due to Bayes is extremely important, and in order to facilitate the appreciation of this a few illustrative applications of Bayes rule will be given in the following.

Example 1

A reinforced concrete beam is considered. From experience it is known that the probability that corrosion of the reinforcement has initiated is P(CI) = 0.01. However, in order to know more precisely an inspection method (non-destructive) has been developed.
The quality of the inspection method may be characterized by the probability that the inspection method will indicate initiated corrosion given that corrosion has initiated $P(I|CI)$ (the probability of detection or likelihood of an indication given corrosion initiation) and the probability that the inspection method will indicate initiated corrosion given that no corrosion has initiated $P(I|\overline{CI})$ (the probability of erroneous findings or the likelihood of an indication given no corrosion initiation).

For the inspection method at hand the following characteristics have been established

$P(I|CI)$ =0.8



$P(I|\overline{CI})$ =0.1

An inspection of the concrete beam is conducted with the result that the inspection method indicates that corrosion has initiated. The question is now, based on the findings from the inspection what is the probability that corrosion of the reinforcement has initiated.

The answer is readily found by application of Bayes rule (see also Figure 3)

$$P(CI|I) = \frac{P(I|CI)\cdot P(CI)}{P(I|CI)\cdot P(CI) + P(I|\overline{CI})\cdot P(\overline{CI})} = \frac{P(I \cap CI)}{P(I)} \tag{17}$$

Figure 3 Illustration of the event tree for inspection and corrosion events.

The probability of obtaining an indication of corrosion at the inspection is

$$P(I) = P(I|CI)\cdot P(CI) + P(I|\overline{CI})\cdot P(\overline{CI}) = 0.8\cdot 0.01 + 0.1\cdot(1-0.01) = 0.107$$

The probability of achieving an indication of corrosion and at the same time to have corrosion initiated is

$$P(I \cap CI) = P(I|CI) \cdot P(CI) = 0.8 \cdot 0.01 = 0.008$$

Thus the probability that corrosion of the reinforcement has initiated given an indication of corrosion at the inspection is

$$P(CI|I) = \frac{0.008}{0.107} = 0.075$$

In fact the probability of initiated corrosion given an indication of corrosion is surprising low. This is mainly due to the relatively high probability of an erroneous indication of corrosion at the inspection, i.e. the inspection method is not sufficiently good.

Example 2

An old reinforced concrete bridge is reassessed in connection with an upgrading of the allowable traffic, see also Schneider (1995). The concrete compression strength class is unknown but concrete cylinder samples may be taken from the bridge and tested in the laboratory.
The following classification of the concrete is assumed

B1: $0 \le \sigma_c < 30$

B2: $30 \le \sigma_c < 40$

B3: $40 \le \sigma_c$

Even though the concrete class is unknown experience with similar bridges suggests that the probability of the concrete of the bridge belongs to class B1, B2 and B3 is 0.65, 0.24 and 0.11 respectively. This information comprise the prior information – prior to any experiment result.

The test method is not perfect in the sense that even though the test indicates a value of the concrete compression strength belonging to a certain class there is a certain probability that the concrete belongs to another class. The likelihoods for the considered test method are given in Table 1.

It is assumed that one test is performed and a concrete compression strength equal to 36.2 MPa i.e. in the interval of class B2 is found.

Using Bayes rule the probability the concrete belongs to the different classes may now be updated, e.g. the posterior probability that the concrete belongs to class B2 is given by

$$P(B2|I = B2) = \frac{0.61 \cdot 0.24}{0.61 \cdot 0.24 + 0.28 \cdot 0.65 + 0.32 \cdot 0.11} = 0.40$$

The posterior probabilities for the other classes may be calculated in a similar manner, the results are given in Table 1.

Concrete Grade	Prior Probability	Likelihood $P(I\|Bi)$			Posterior probabilities
		I = B1	I = B2	I = B3	
B1	0.65	0.71	0.28	0.01	0.50
B2	0.24	0.18	0.61	0.21	0.40
B3	0.11	0.02	0.32	0.66	0.10

Table 1 Summary of prior probabilities, likelihood's of experiment outcomes and posterior probabilities given one test result in the interval of class B2.

2 Descriptive Statistics

2.1 Introduction

In order to assess the characteristics and the level of uncertainty of a given quantity of interest one of the first steps is to investigate the data available, such as observations and test results. For this purpose the descriptive statistics is an adequate means. The descriptive statistics do not assume anything in terms of the degree or nature of the randomness underlying the data analyzed but is merely a convenient tool to reduce the date to a manageable form suitable for further analysis.

2.2 Graphical representations

Graphical representations are convenient as they show the main characteristics of the available data in a condensed and visual format. Among the different possibilities of graphical representations the histograms frequency distributions and cumulative frequency distributions are the most commonly used and these will thus be shown by an example in the following.

Consider the data in Table 2 on the concrete cube compression strength as obtained by laboratory experiments. In the table the data are given unordered and ordered respectively.

In Table 3 the data have been further processed and the observed range of concrete cube compression strengths has been divided into intervals. For each interval the interval mid point has been determined and the number of observations observed within each interval has been evaluated. Finally the frequency of the observations of concrete cube compression strengths within each interval has been evaluated and by summing these up over the intervals the cumulative frequencies are calculated, see Table 3.

Histograms may now be constructed by plotting the number of observed values within a certain interval against the intervals as shown in Figure 4. In Figure 4 also the frequency distribution is shown giving the relative number of observations of the concrete cube compression strengths for the different intervals. If the scale of the frequencies in the frequency distribution plot were divided by the interval length a frequency density plot would have been obtained.

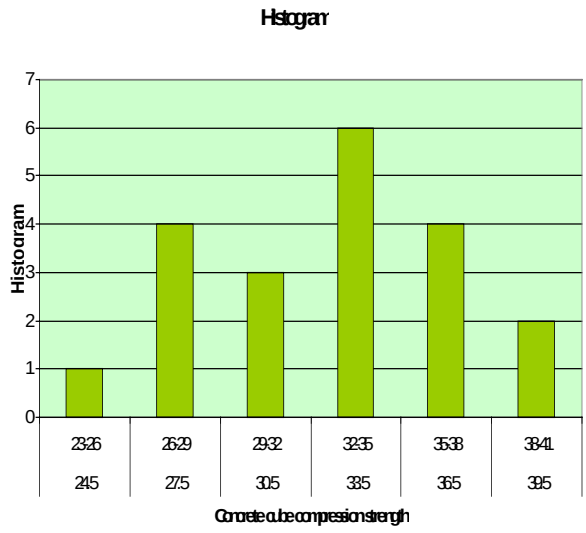
The area under the corresponding curve would then have been unity and this representation is also called a sample probability density function.

Concrete cube compression strengths (MPa)	
Unordered	Ordered
35.8	24.4
39.2	27.6
34.6	27.8
27.6	27.9
37.1	28.5
33.3	30.1
32.8	30.3
34.1	31.7
27.9	32.2
24.4	32.8
27.8	33.3
33.5	33.5
35.9	34.1
39.7	34.6
28.5	35.8
30.3	35.9
31.7	36.8
32.2	37.1
36.8	39.2
30.1	39.7

Table 2 Concrete cube compression strength experiment results.

Class midpoint	Interval	Number of observations	Frequency	Commulative frequency
24.5	23-26	1	0.05	0.05
27.5	26-29	4	0.2	0.25
30.5	29-32	3	0.15	0.4
33.5	32-35	6	0.3	0.7
36.5	35-38	4	0.2	0.9
39.5	38-41	2	0.1	1

Table 3 Tabularized summary of the observed concrete cube compression strengths.



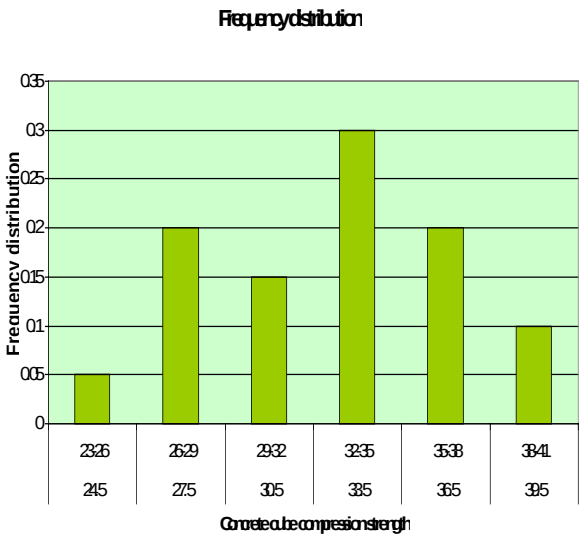


Figure 4 Histogram and frequency distribution representations of the observed concrete cube compression strengths.

Finally in Figure 5 the cumulative frequencies are plotted against the intervals.

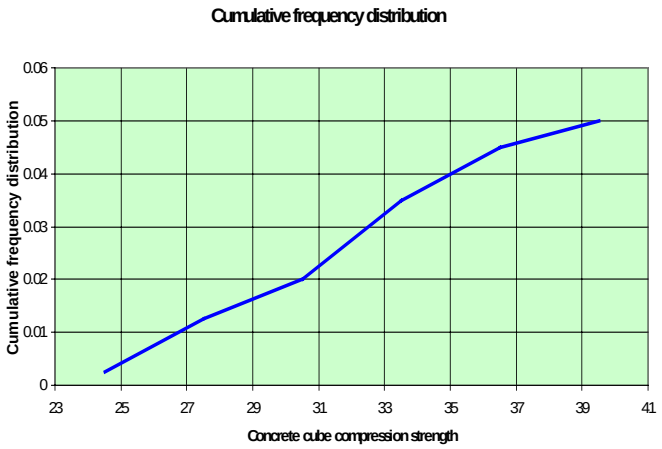


Figure 5 Cumulative frequency plot of the observed concrete cube compression strengths.

It should be noted that the interval width might have a significant effect on the appearance of the data. No general guidelines can be given on the choice of the interval width but in Benjamin and Cornell [1] it is suggested to split the interval between the max and the min value into k intervals where k is given by

$$k = 1 + 3.3 \log n \tag{18}$$

where n is the number of data points in the data set.

2.3 Numerical summaries

2.3.1 Central measures

One of the most useful numerical summaries is the sample mean. If the data set is collected in the vector $\mathbf{x} = (x_1, x_2, ..., x_3)^T$ the sample mean $\bar{x}$ is simply given as

$$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i \tag{19}$$

Considering the data set of the concrete cube compression strengths the sample mean is readily evaluated using Equation (19) to 32.67 MPa.

The sample mean may be interpreted as a central value of the data set. If on the basis of the data set one should give only one value characterizing the concrete cube compression strength one would normally use the sample mean. Other central measures are the mode of the data set i.e. the most frequently occurring value in the data set and the median of the data set, i.e. the middle value in the ordered list if n is odd or the average value of the two middle values if n is even.

Referring to the data set of concrete cube compression strengths the mode would thus be 33.5 MPa and the median would be 31.5 MPa.

2.3.2 Dispersion measures

The variability or the dispersion of the data set is also an important characteristic of the data set. This dispersion may be characterized by the sample variance s^2 given by

$$s^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 \tag{20}$$

and the sample standard deviation s is defined as the positive square root of the sample variance.
The sample variance thus is mean of the squared deviations from the sample mean and is in this way analogous to the moment of inertia.
The sample variance for the concrete cube compression strengths may be evaluated using Equation (20) and is found to be 17.22 MPa and the sample standard deviation thus 4.14 MPa.

As a means of comparison of the between the dispersions of different data sets the dimensionless sample coefficient of variation v is convenient. The sample coefficient of variation v is defined as the ratio of the sample standard deviation to the sample mean, i.e. given by

$$v = \frac{s}{\bar{x}} \tag{21}$$

For the considered concrete cube compression strength data the sample coefficient of variation may thus be evaluated to be 0.13.

2.3.3 Other measures

Whereas the sample mean is a central measure of a data set and the sample variance a measure of the dispersion around the sample mean it is also useful to have some characteristic indicating the degree of symmetry of the data set. To this end the sample coefficient of skewness, which is a simple logical extension of the sample variance is suitable. The sample coefficient of skewness η is defined as

$$\eta = \frac{1}{n}\cdot\frac{\sum_{i=1}^{n}(x_i - \bar{x})^3}{s^3} \tag{22}$$

This coefficient is positive for histograms skewed to the right and negative for those skewed to the left. For the concrete cube compression strengths the sample coefficient of skewness is – 0.11.

In a similar way the sample coefficient of kurtosis κ is defined as

$$\kappa = \frac{1}{n}\cdot\frac{\sum_{i=1}^{n}(x_i - \bar{x})^4}{s^4} \tag{23}$$

which is a measure of the peakedness of the histogram. Typically one would compare the sample coefficient of kurtosis to that of a normal distribution, which is 3.0. The kurtosis for the concrete cube compression strength is evaluated to 2.02, i.e. the considered data set is less peaked than the normal distribution.

2.3.4. Measures of correlation

Observations are often made of two characteristics simultaneously as shown in Figure 3 where two examples of so-called scattergrams are illustrated.

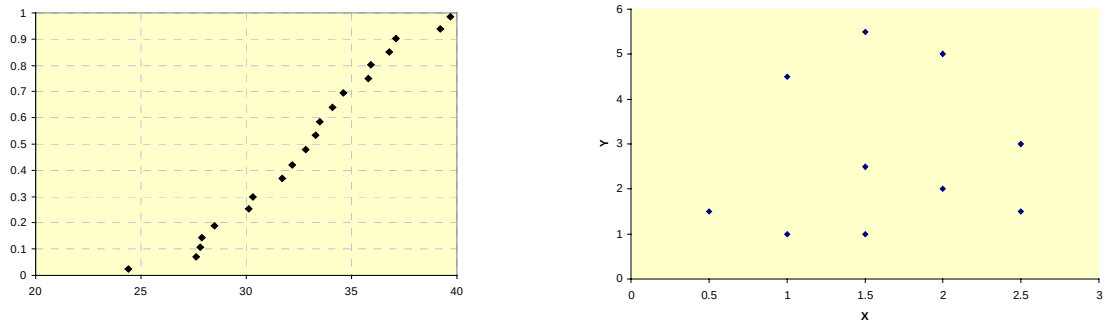


Figure 6 Two examples of paired data sets.

In the scattergram to the left in Figure 3 there appear to be little dependency between the observed data pairs whereas the opposite is evident in the example to the right.
As a characteristic indicating the tendency toward high-high pairings and low-low pairings, i.e. a measure of the correlation between the observed data sets the sample covariance s_{XY} is useful, which is defined as

$$s_{XY} = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})\cdot(y_i - \bar{y}) \tag{24}$$

The sample covariance has the property that if there is a tendency in the data set that the values of x_i and y_i both are higher than $\bar{x}$ and $\bar{y}$ at the same time then most of the terms in the sum will be positive and the sample covariance will be positive. The other way around will result in a negative sample covariance.

The sample covariance may be normalized in respect to the sample standard deviations of the individual data sets s_X and s_Y and the result is called the sample correlation coefficient r_{XY} defined as

$$r_{XY} = \frac{1}{n}\frac{\sum_{i=1}^{n}(x_i - \bar{x})\cdot(y_i - \bar{y})}{s_X \cdot s_Y} \tag{25}$$

The sample correlation coefficient has the property that it is limited to the interval $-1 \le r_{XY} \le 1$ and the extreme values of the interval are only achieved in case the data pairs are perfectly correlated, implying that the points on the scattergram lie on a straight line.
For the example shown in Figure 6 there is almost zero correlation in the left hand side and almost full positive correlation in the right hand side.

3 Uncertainty Modeling

3.1 Uncertainties in Engineering Problems

Probabilities are intended to quantify uncertainties. In risk and reliability analysis the following types of uncertainties are generally distinguished.

- inherent random variability or uncertainty
- uncertainty due to inadequate knowledge (model uncertainty)
- statistical uncertainty.

The inherent random variability's and uncertainties may be split up into two categories: uncertainties, which can, and uncertainties which cannot, be affected by human activities. Many kinds of action parameters, for example, wind speed and earthquake ground motion intensity belong to the second category. The first category concerns, for example, the

uncertainties of strength values and dimensions of concrete. These uncertainties can be reduced by the use of more advanced production and quality control methods, which, on the other hand, may cause an increase of production costs.

The uncertainties due to inadequate knowledge can also be subdivided into two categories. To one of these categories belong, for example, the uncertainties of physical models or for which knowledge can be increased - and thus the uncertainty can be decreased - by research or other similar activities. Also measurement errors belong to this category of uncertainties. To the other category belongs, for example, uncertainties, which depend on future development. One example may be the future development of the traffic loads on road bridges or the increase in the water level of the oceans. The possibility to decrease these uncertainties by research or similar activities is very limited.

The statistical uncertainties are associated with the statistical evaluation of results of tests or observations. They may result from:

- a limited number of observations or test results which cause uncertainties in the estimation of statistical parameters, e.g. mean and standard deviation
- neglecting systematic variations of the observed variables e.g. of climate variables
- neglecting possible correlations.

A typical example of statistical uncertainties is also the uncertainty associated with the frequentistically assessed failure probabilities of components for which only a limited number of observed failures are available. In such cases it is of utmost importance to take into account the statistical uncertainty, something, which is often not done or not done consistently.

The statistical uncertainties can normally be decreased by increasing test and observational efforts.

In risk and reliability analysis all three types of uncertainty need to be considered. This also indicates the need for a Bayesian interpretation of probability.

The above descriptions of uncertainties highlights that some uncertainties are unavoidable (the wind velocity for the next year) while some others might be removed at the cost of some expense. In other words: uncertainty is in some cases voluntary.

3.2 Random variables

The performance of an engineering system, facility or installation (in the following referred to as system) may usually be modeled in mathematical physical terms in conjunction with empirical relations.

For a given set of model parameters the performance of the considered system can be determined on the basis of this model. The basic random variables are defined as the parameters that carry the entire uncertain input to the considered model.

The basic random variables must be able to represent all types of uncertainties that are included in the analysis. The uncertainties, which must be considered are as previously mentioned the physical uncertainty, the statistical uncertainty and the model uncertainty. The physical uncertainties are typically uncertainties associated with the loading environment, the geometry of the structure, the material properties and the repair qualities. The statistical uncertainties arise due to incomplete statistical information e.g. due to a small number of materials tests. Finally, the model uncertainties must be considered to take into account the uncertainty associated with the idealized mathematical descriptions used to approximate the actual physical behavior of the structure.

Modern methods of reliability and risk analysis allow for a very general representation of these uncertainties ranging from non-stationary stochastic processes and fields to time-invariant random variables, see e.g. Melchers [2]. In most cases it is sufficient to model the uncertain quantities by random variables with given distribution functions and distribution parameters estimated on basis of statistical and/or subjective information. Therefore the following is concerned with a basic description of the characteristic of random variables.

A random variable, which can take on any value is called a continuous random variable. The probability that such a random variable takes on a specific value is zero. The probability that a continuous random variable, X, is less than a value, x, is given by the probability distribution function

$$F_X(x) = P(X < x) \tag{26}$$

In general large letters denote a random variable and small letters denote an outcome on a stochastic variable. An example of a continuous probability distribution function is illustrated in Figure 7.

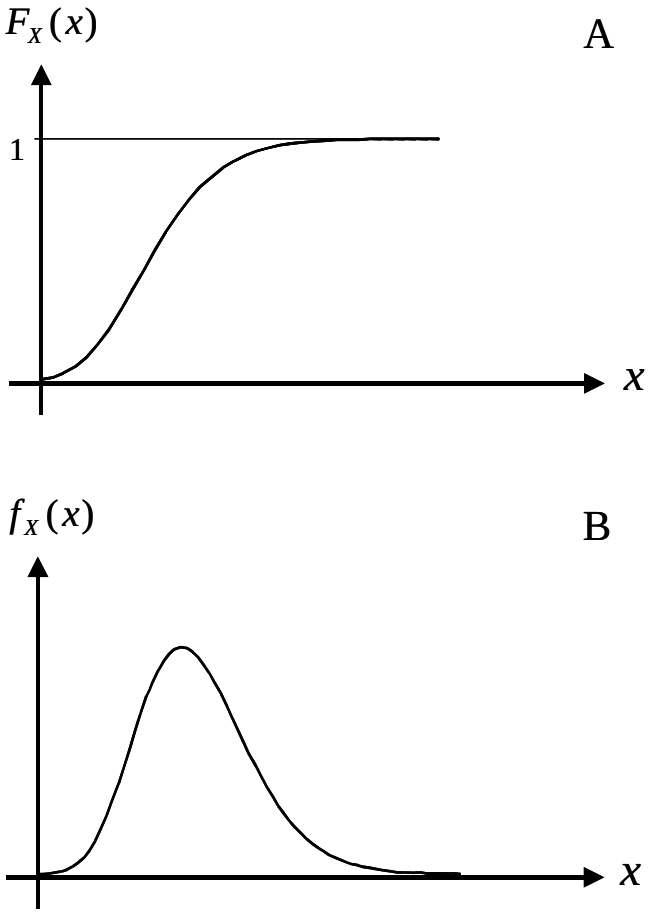


Figure 7 Illustration of a probability distribution function A) and a probability density function B) for a continuous random variable.

For continuous random variables the probability density function is given by

$$f_X(x) = \frac{\partial F(x)}{\partial x} \tag{27}$$

The probability of an outcome in the interval [x;d+dx] where dx is small is given by $P(x \in [x; x+dx]) = f_X(z)dx$.

Random variables with a finite or infinite countable sample space are called discrete random variables. For discrete random variables the probability distribution function is given as

$$P_X(x) = \sum_{x_i < x} p_X(x_i) \tag{28}$$

where $p_X(x_i)$ is the probability density function given as

$$p_X(x_i) = P(X = x) \tag{29}$$

A discrete probability distribution function and probability density function is illustrated in Figure 8.

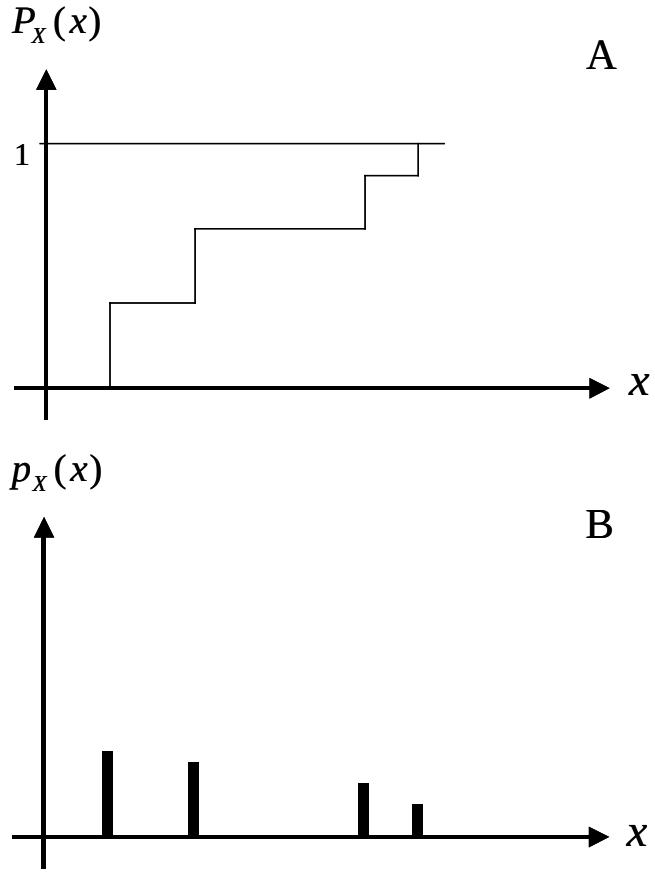


Figure 8 Illustration of a probability distribution function A) and a probability density function B) for a discrete random variable.

Probability distribution functions may be defined in terms of the parameters or moments. Often probability distribution functions and probability density functions are written as $F_X(x,\mathbf{p})$ and $f_X(x,\mathbf{p})$ respectively to indicate the parameters (or moments) defining the functions.

The moments of a random variable are defined by

$$m_i = E\left[X^i\right] = \int_{-\infty}^{\infty} x^i \cdot f_X(x)dx \tag{30}$$

The mean of a stochastic variable, X , is defined as the first moment, i.e.

$$\mu_X = E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)dx \tag{31}$$

and the standard deviation, σ_X , is defined by the second central moment, i.e.

$$\sigma_X{}^2 = \mathrm{Var}[\mathrm{X}] = \int_{-\infty}^{\infty} (x-\mu_X)^2 \cdot f_X(x)dx \tag{32}$$

where $\mathrm{Var}[\mathrm{X}]$ is called the variance of X .

Whether the distribution and density functions are defined by their moments or by parameters is a matter of convenience and it is always possible to establish the one from the other.

As an example consider a continuous random variable with a uniform (constant) probability density function in the interval $[a;b]$ as illustrated in Figure 9.

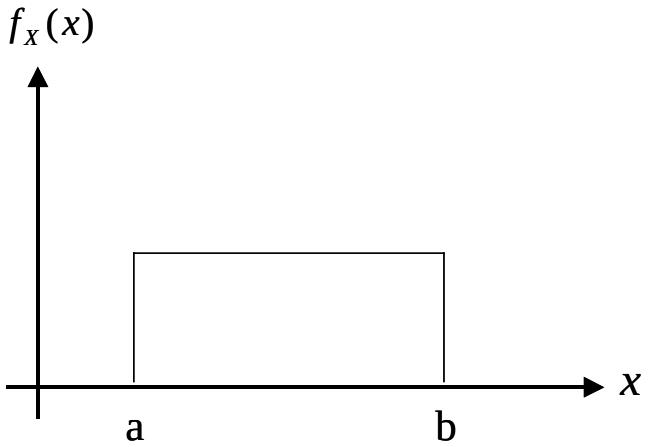


Figure 9 Continuous random variable with uniform distribution function.

The probability density function for a uniformly distributed random variable is easily seen to be

$$f_X(x) = \begin{cases} 0, & x < a \\ \dfrac{1}{b-a}, & a \le x \le b \\ 0, & b < x \end{cases} \tag{33}$$

remembering that the area under the probability density function must integrate to 1. In Equation (33) a and b are the parameters of the probability density function.

The probability distribution for a uniformly distributed random variable is thus

$$F_X(x) = \begin{cases} 0, & \text{x} < \text{a} \\ \displaystyle\int_a^x f_X(y)dy = \int_a^x \frac{1}{b-a}dy = \frac{(x-a)}{(b-a)}, & \text{a} \le \text{x} \le \text{b} \\ 1, & \text{b} < \text{x} \end{cases} \tag{34}$$

The first moment i.e. the mean value (see Equation (31)) of a continuous random variable with uniform distribution is thus

$$\mu_X = E[X] = \int_a^b x \cdot f_X(x)dx = \int_a^b \frac{x}{b-a}dx = \frac{x^2}{2(b-a)}\Bigg|_a^b = \frac{(b+a)}{2} \tag{35}$$

and the standard deviation σ_X (see Equation (32)) is given through the second central moment

$$\sigma_X^2 = E\left[(X-\mu)^2\right] = \int_a^b (x-\mu)^2 \cdot f_X(x)dx = \int_a^b \frac{(x-\mu)^2}{(b-a)}dx = \left.\frac{\frac{1}{3}x^3 - x^2\mu + x\mu^2}{(b-a)}\right|_a^b$$
$$= \frac{1}{12}(b-a)^2 \tag{36}$$

If an n-dimensional vector of continuous random variables $\mathbf{X} = (X_1, X_2, \ldots, X_n)$, is considered the joint distribution function is given by

$$F_{\mathbf{X}}(\mathbf{X}) = P(X_1 \le x_1 \cap X_2 \le x_2 \cap \ldots \cap X_n \le x_n) \tag{37}$$

and the probability density function is

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{\partial^n}{\partial z_1 \partial z_2 \ldots \partial z_n} F_{\mathbf{X}}(\mathbf{x}) \tag{38}$$

The covariance between X_i and X_j is defined by

$$C_{X_i X_j} = E\left[(X_i - \mu_{X_i})(X_j - \mu_{X_j})\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x_i - \mu_{X_i})(x_j - \mu_{X_j}) f_{X_i X_j}(x_i, x_j)dx_i dx_j \tag{39}$$

and is also called the mixed central moment between the variables X_i and X_j.

The covariance expresses the dependence between two variables. It is evident that $C_{X_i X_i} = \mathrm{Var}[\mathrm{X_i}]$. On the basis of the covariance the correlation coefficient is defined by

$$\rho_{X_i X_j} = \frac{C_{X_i X_j}}{\sigma_{\mathrm{X_i}} \sigma_{X_j}} \tag{40}$$

It is seen that $\rho_{X_i X_i} = 1$. The correlation coefficients can only take values in the interval [-1;1]. A negative correlation coefficient between two variables implies that if the outcome of one variable is large compared to its mean value the outcome of the other variable is likely to be small compared to its mean value. A positive correlation coefficient between two variables implies that if the outcome of one variable is large compared to its mean value the outcome of the other variable is also likely to be large compared to its mean value. If two variables are independent their correlation coefficient is zero and the joint density function is the product of the 1-dimensional density functions. In many cases it is possible to obtain a sufficiently accurate approximation to the n-dimensional distribution function from the 1-dimensional distribution functions of the n variables and their parameters, and the correlation coefficients.

The conditional probability density function for the random variable X_1, conditional on the out come of the random variable X_2 is denoted $f_{X_1|X_2}(x_1|x_2)$ and defined by

$$f_{X_1|X_2}(x_1|x_2) = \frac{f_{X_1,X_2}(x_1, x_2)}{f_{X_2}(x_2)} \tag{41}$$

in accordance with the definition of conditional probability given previously.
Given the n-dimensional density function of the variables $\mathbf{X}$ it is possible to determine the probability of obtaining an outcome of $\mathbf{X}$ that belongs to a given region, $[\Omega_F]$, is

$$P(\mathbf{X} \in [\Omega_F]) = \int_{\Omega_F} f_{\mathbf{X}}(\mathbf{X}) d\mathbf{x} \tag{42}$$

In Table 4 a selection of probability density functions is given with the definition of distribution parameters and moments.

The most commonly used probability models in reliability calculations are the:

- normal distribution
- lognormal distribution
- extreme value distributions

The normal distribution is used if the stochastic variable is composed as a sum of large number of individual contributions (Central limit theorem). A lognormal distribution follows if log(X) is normally distributed. As a result a lognormal distributed variable is always non-negative, which makes it appropriate for the modeling of non-negative uncertain variables.

The extreme distributions form a complete family for cases when a variable is the maximum or minimum of e.g. a stochastic process over a given time interval or a large number of variables. Typical examples are the Gumbel, the exponential, the Rayleigh, the Weibull, and the Frechet distribution.

Distribution type	Para-meters	Moments
Rectangular $a \le x \le b$ $f_x(x) = \frac{1}{b-a}$	a b	$\mu = \frac{a+b}{2}$ $\sigma = \frac{b-a}{\sqrt{12}}$
Normal $f_x(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)$	μ $\sigma > 0$	μ σ
Lognormal $x > 0$ $f_x(x) = \frac{1}{X\zeta\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\ln x - \lambda}{\zeta}\right)^2\right)$	λ $\zeta > 0$	$\mu = \exp\left(\lambda + \frac{\zeta^2}{2}\right)$ $\sigma = \exp\left(\lambda + \frac{\zeta^2}{2}\right)\sqrt{\exp(\zeta^2) - 1}$
Shifted Lognormal $x > \varepsilon$ $f_x(x) = \frac{1}{(x-\varepsilon)\zeta\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\ln(x-\varepsilon) - \lambda}{\zeta}\right)^2\right)$	λ $\zeta > 0$ ε	$\mu = \varepsilon + \exp\left(\lambda + \frac{\zeta^2}{2}\right)$ $\sigma = \exp\left(\lambda + \frac{\zeta^2}{2}\right)\sqrt{\exp(\zeta^2) - 1}$
Distribution type	Para-meters	Moments
Shifted Exponential $x \ge \varepsilon$ $f_x(x) = \lambda \exp(-\lambda(x-\varepsilon))$	ε $\lambda > 0$	$\mu = \varepsilon + \frac{1}{\lambda}$ $\sigma = \frac{1}{\lambda}$
Gamma $x \ge 0$	$p > 0$ $b > 0$	$\mu = \frac{p}{b}$

$f_X(x) = \frac{b^p}{\Gamma(p)} \exp(-bx) x^{p-1}$		$\sigma = \frac{\sqrt{p}}{b}$
Beta $a \le x \le b,\ r, t \ge 1$ $f_X(x) = \frac{(x-a)^{r-1}(b-x)^{t-1}}{(b-a)^{r+t-1} B(r,t)}$	a b r > 1 t > 1	$\mu = a + (b-a)\frac{r}{r+t}$ $\sigma = \frac{b-a}{r+t}\sqrt{\frac{rt}{r+t+1}}$
Gumbel (L) $-\infty < x < +\infty$ $f_X(x) = \alpha \exp(-\alpha(x-u) - \exp(-\alpha(x-u)))$	u $\alpha > 0$	$\mu = u + \frac{0.577216}{\alpha}$ $\sigma = \frac{\pi}{\alpha\sqrt{6}}$
Frechet (L) $\varepsilon \le x < +\infty,\quad u, k > 0$ $f_X(x) = \frac{k}{u-\varepsilon}\left(\frac{x-\varepsilon}{u-\varepsilon}\right)^{-k-1} \exp\left(-\left(\frac{x-\varepsilon}{u-\varepsilon}\right)^{-k}\right)$	u > 0 k > 0 ε	$\mu = \varepsilon + (u-\varepsilon)\Gamma\left(1-\frac{1}{k}\right)$ $\sigma = (u-\varepsilon)\sqrt{\Gamma\left(1-\frac{2}{k}\right) - \Gamma^2\left(1-\frac{1}{k}\right)}$
Weibull (S) $\varepsilon \le x < +\infty,\quad u, k > 0$ $f_X(x) = \frac{k}{u-\varepsilon}\left(\frac{x-\varepsilon}{u-\varepsilon}\right)^{-k-1} \exp\left(-\left(\frac{x-\varepsilon}{u-\varepsilon}\right)^{-k}\right)$	u > 0 k > 0 ε	$\mu = \varepsilon + (u-\varepsilon)\Gamma\left(1+\frac{1}{k}\right)$ $\sigma = (u-\varepsilon)\sqrt{\Gamma\left(1+\frac{2}{k}\right) - \Gamma^2\left(1+\frac{1}{k}\right)}$

Table 4 Probability distributions, Schneider [3].

3.3 Stochastic processes

Random quantities may be time variant in the sense that they have different realizations at different times, examples hereof are the wind velocity, wave heights, snow fall and water levels. Such phenomena may in some cases be described in terms of random variables e.g. for the modeling of the point in time intensity of the wind velocity, or the maximum wind velocity during one year. However, in many cases this is not possible and then it is necessary to consider the random phenomena as a random process.

A stochastic process $X(t)$ is as mentioned a random function of time meaning that for any point in time the value of X is a random variable. A realization of a random process (water level variation) is illustrated in Figure 10.

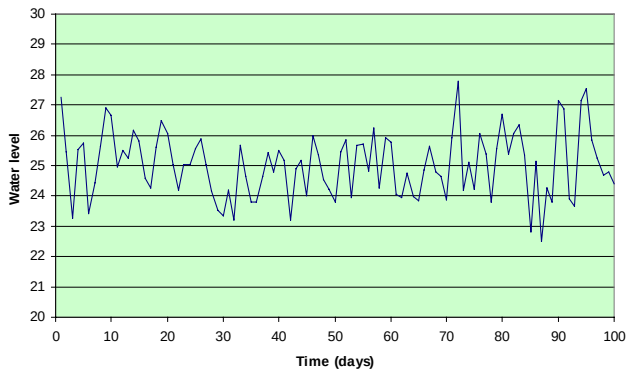


Figure 10 Realisation of the water level variation as function of time.

In accordance with the definition of the mean value of a random variable the mean value of all the possible realization of the stochastic process at time t is given by

$$\mu_X(t) = \int_{-\infty}^{\infty} x \cdot f_X(x,t)dx \tag{43}$$

The correlation between all possible realizations at two points in time t_1 and t_2 is described through the so-called autocorrelation function $R_{XX}(t_1,t_2)$. Auto means that the function refers to only one realization. The autocorrelation function is defined by

$$R_{XX}(t_1,t_2) = E[X(t_1)X(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 \cdot x_2 \cdot f_{XX}(x_1,x_2;t_1,t_2)dx_1dx_2 \tag{44}$$

The auto-covariance function is defined as

$$\begin{aligned} C_{XX}(t_1,t_2) &= E[(X(t_1)-\mu_X(t_1))(X(t_2)-\mu_X(t_2))] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x_1-\mu_X(t_1))\cdot(x_2-\mu_X(t_2))\cdot f_{XX}(x_1,x_2;t_1,t_2)dx_1dx_2 \end{aligned} \tag{45}$$

for $t_1 = t_2 = t$ the auto-covariance function becomes the covariance function

$$\sigma_X^2(t) = C_{XX}(t,t) = R_{XX}(t,t) - \mu_X^2(t) \tag{46}$$

where $\sigma_X(t)$ is the standard deviation function.

The above definitions for the scalar process $X(t)$ may be extended to cover also vector valued processes $\mathbf{X}(t) = (X_1(t), X_2(t),.., X_n(t))^T$ having covariance functions $C_{X_iX_j} = \mathrm{cov}[X_i(t_1), X_j(t_2)]$. For $i = j$ these become the autocovariance functions and when $i \neq j$ these are termed the cross-covariance functions. Finally the correlation function may be defined as

$$\rho[X_i(t_1), X_j(t_2)] = \frac{\mathrm{cov}[X_i(t_1), X_j(t_2)]}{\sigma_{X_i}(t_1)\cdot\sigma_{X_j}(t_2)} \tag{47}$$

Typically the correlation function is an exponentially decaying function in time.

Having defined the mean value functions and the cross-correlation functions for the stochastic process $\mathbf{X}(t)$ the probability that the process remains within a certain domain $\mathbf{D}$ in the time interval $[0,t]$ may be evaluated by

$$P_f(t) = 1 - P(N(t) = 0 \,|\, \mathbf{X}(0) \in \mathbf{D}) \cdot P(\mathbf{X}(0) \in \mathbf{D}) \tag{48}$$

where N(t) is the number of out-crossings of the stochastic process out of the domain $\mathbf{D}$ in the time interval $[0,t]$. The solution of Equation (48) is in general rather difficult to obtain and will not be treated any further in the present section.

3.3.1 Stationary and ergodicity

When the mean value function $\mu_X(t)$ and the autocorrelation function $R_{XX}(t)$ of a stochastic process $X(t)$ do not depend on time the process is said to be weakly stationary. Only if all the moments of a stochastic process are independent of time the stochastic process is said to be strictly stationary.
A consequence of stationarity is that the auto-covariance functions and autocorrelation function only depend on the time difference $\tau = t_1 - t_2$. In this case Equation (44) may be written as

$$R_{XX}(\tau) = E\left[X(t)X(t+\tau)\right] \tag{49}$$

It should be noted that for weakly stationary normal stochastic processes the requirements for strict stationarity are automatically fulfilled as the normal distribution function is completely defined by the first two moments.

Stationarity in principle implies that the process cannot start or stop, however, for practical purposes this requirement may be released if the process is considered a sufficient time after its start and/or before its end. Also stationarity may be assumed even for slowly varying stochastic processes if sufficiently short time intervals are considered.

If in addition to stationarity the mean value function and the autocorrelation function of a stochastic process may be defined by a time average over one realization of the stochastic process the process is said to be ergodic. If all moments of the process may be defined in this way the process is said to be strictly ergodic.

The assumption of ergodicity is especially important for the estimation of the statistical characteristics of stochastic processes when only one (or a few sufficiently long) realization of the process is available. In practice ergodicity is in such cases often assumed unless of cause evidence indicate the contrary.

3.4 Modeling of loads

In the following the term load will be related to forces acting on structural components and systems, but the notions and concepts introduced will to a large extent be valid for other types of influences, such as temperature, aggressive chemicals and radiation “acting from the outside” of the engineering system of consideration.

3.4.1 Modeling of individual loads

Loads and/or load effects are uncertain due to

- Random variations in space and time
- Model uncertainties

Whereas the model uncertainties associated with the physical model used to represent the loads and/or load effects in the reliability analysis may be represented by random variables as explained in a later section the loads themselves are usually time and space varying quantities and thus best modeled by stochastic processes.

For loading on structures it is helpful to categorize loads according to their time varying characteristics.

- Permanent or variable
- Fixed or free
- Static or dynamic

whereby their nature in regard to variability of magnitude in time, variability in regard to position in time and their nature in regard to their effect on the engineering system may be characterized. Knowing these characteristics is a prerequisite for the probabilistic modeling.

As an example consider the case where the reliability in regard to ultimate collapse of a reservoir structure is analyzed. The load acting on the structure considered is the hydrostatic pressure due to the water contained in the reservoir. As the hydro static pressure varies with the water level in the reservoir, which is dependent on the amount of rainwater flowing into the reservoir the loading must be considered to be variable in time. Furthermore due to the characteristics of the hydrostatic pressure loading the loading is assumed fixed in space. Finally as the reliability of the reservoir structure is considered in regard to an ultimate collapse failure mode and is dynamically insensitive the loading may be idealized as acting static.

Having identified the characteristics of the considered load the probabilistic modeling may proceed by

- précising the definition of the random variables used to represent the uncertainties in the loading
- selecting a suitable distribution type to represent the random variable
- to assign the distribution parameters of the selected distribution.

The first point is in practice usually the most important issue to solve and requires a clear definition of the reliability problem to be considered. For the example considered the random variable should only be concerned about the modeling of the random variations in time of the reservoir water level.
The second point can as mentioned in the previous section not be performed on the basis of data alone but moreover requires the simultaneous consideration of data, experience and physical understanding. However, in the following some guidelines are given to assist in the choice of distribution types.

Permanent loads
Typically permanent loads on a structural system are composed as a sum of self-weights of many individual parts of the structure. As the sum of a (even moderately) large number of uncertain variables, under the condition that none of the individual terms are dominating, converges towards the normal distribution this distribution is often a good choice for permanent loads.

Variable loads
For continuous time-varying loads, which can be described by one quantity, i.e. the water level in the considered example one can define a number of related probability distribution functions. Often the simplest, namely the arbitrary point in time distribution is considered.
If $x(t^*)$ is a realisation of a single time-varying load at time t^* then $F_X(x)$ is the arbitrary point in time probability distribution function of $X(t)$ defined by

$$F_X = P(X(t^*) \leq 0) \tag{50}$$

In Figure 11 a time series of daily measurements of the maximum water level are given together with the histograms showing the sampling frequency distribution of the daily maximal water level, i.e. the arbitrary point in time frequency distribution and the frequency distribution of the 10 days maximal water levels.

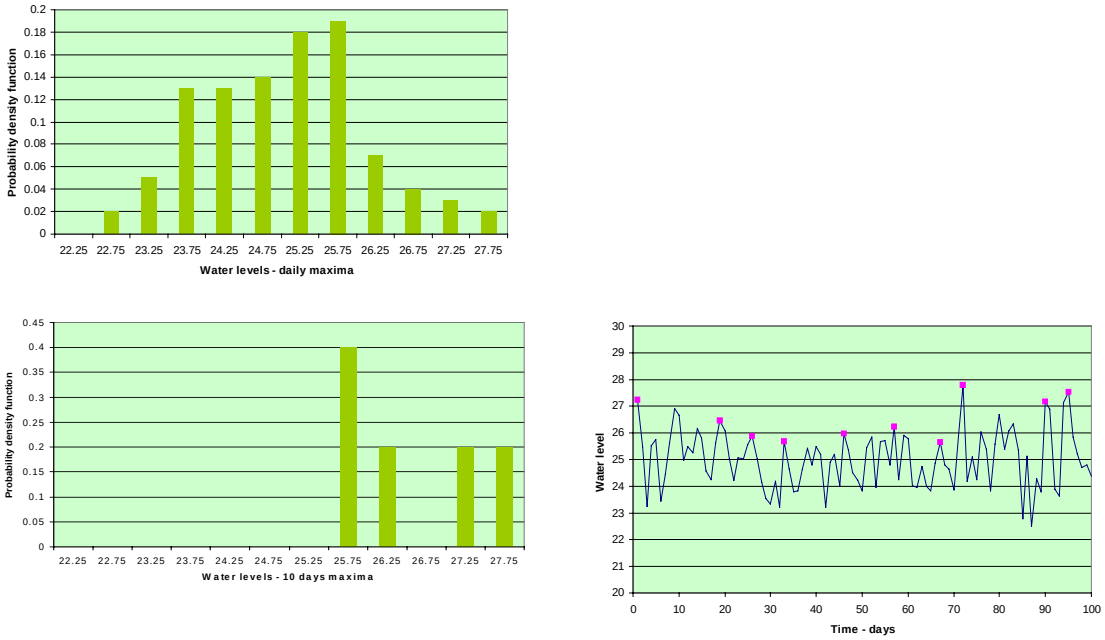


Figure 11 Time series of recorded daily maximal water levels together with the daily (arbitrary point in time) and 10 days maximal frequency distributions of water levels.

When extreme events are of interest the arbitrary point in time distribution of the load variable is not of immediate relevance but rather the distribution of the maximal values of the considered quantity over a given reference period. From Figure 11 it is seen that these distributions may differ significantly.

If the loading process $X(t)$ may be assumed to be ergodic the distribution of the largest extreme in a reference period T, $F_{X,T}^{\max}(x)$ can be thought of as being generated by sampling values of the maximal realization $x_{\max}$ from successive reference periods T. If the values of $x_{\max}$ are represented by the random variable Y, the probability distribution function $F_Y(y)$ is the probability distribution function of the extreme maximum loading corresponding to the considered reference period T.

In the same way the probability distribution of the largest extreme load in a period of $n \cdot T$, $F_{X,n\cdot T}^{\max}(x)$ may be determined from the probability distribution function of the largest extreme in the period T, $F_{X,T}^{\max}(x)$, by

$$F_{X,n\cdot T}^{\max}(x) = (F_{X,T}^{\max}(x))^n \tag{51}$$

which follows from the multiplication law for independent events.

For extreme loads reference is often made to the return period. The return period for a load event may be defined by

$$T_R = n \cdot T = \frac{1}{(1 - F_{X,T}^{\max}(x))} \tag{52}$$

where T is the reference period for the distribution of the extreme events $F_{X,T}^{\max}(x)$.

If as an example the annual probability of an extreme load event is 0.02 the return period for this load event is 50 years.

Similarly to the derivation of Equation (51) the probability distribution for the extreme minimum value in a considered reference period T, $F_{X,n\cdot T}^{\min}(x)$ may be found as

$$F_{X,n\cdot T}^{\min}(x) = 1 - (1 - F_{X,T}^{\min}(x))^n \tag{53}$$

In cases as described in the above the maximum load may be modeled probabilistically by one of the extreme value distributions mentioned previously (Weibull, Gumbel, Frechet). However, as mentioned previously it is required that the considered process is ergodic. This requirement may be difficult to fulfill in practical situations but can often be fulfilled approximately by choosing the reference period sufficiently small such that the assumption of ergodicity is "almost" fulfilled.

In the above only continuous time varying load processes were discussed. However, in practical application non-continuous processes such as the various types of Poisson processes also form important tools in the modelling of load processes not least in connection with the modelling of the spatial occurrences of the loads, see e.g. Melchers [2]. A comprehensive library of probabilistic models for loads are given in JCSS [7].

3.4.2 Combinations of loads

One important aspect when considering reliability and risk analysis is the modeling of the extremes of combinations of different loads – the load combination problem.

A typical example of the time variation of different loads acting on structural component or system is shown in Figure 12.

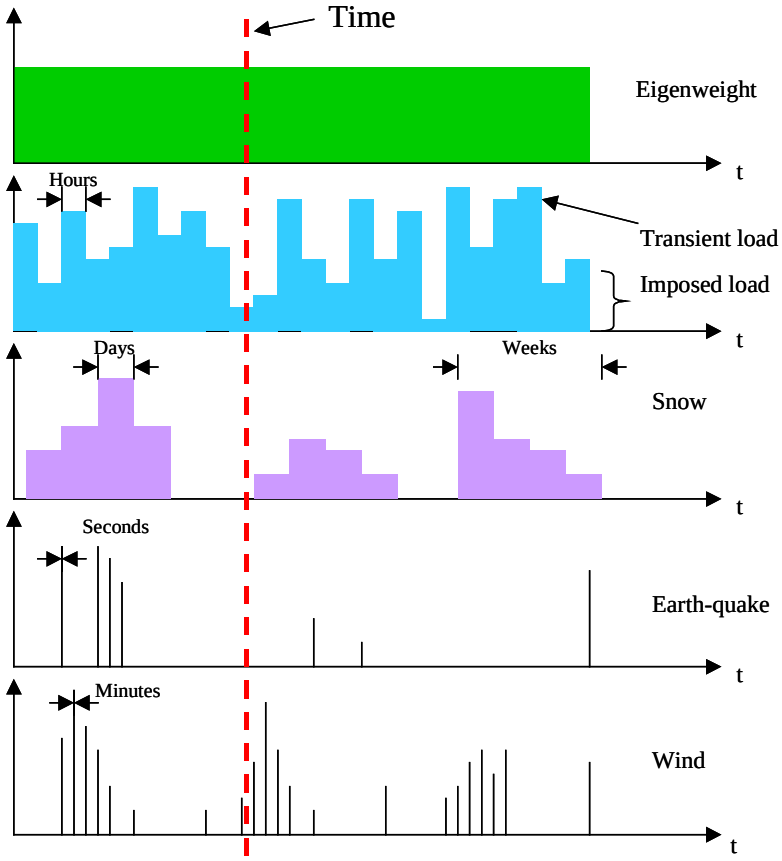


Figure 12 Illustration of the time variation of different loads on a structural component or system.

The maximum load acting on the considered structure within a given reference period T may be assessed through the maximum of the sum of the individually acting loads, i.e.

$$X_{\max}(T) = \max_{T}\{X_1(t) + X_2(t) + ... + X_n(t)\} \tag{54}$$

however, the assessment requires a detailed knowledge about the time variations of the individual loads.

A general solution to equation (54) is hardly achievable but solutions exist for special cases of continuous processes and different types of non-continuous processes, see e.g. Melchers [2] and Thoft-Christensen & Baker [5]. However, approximate solutions to Equation (54) may be established and in the following the most simple and most widely used of these will be described.

Turkstra's load combination rule

By consideration of Figure 12 it is clear that it is highly unlikely (especially when the number of loads is large) that all n loads will attain their maximum at the same time. Therefore it is too conservative to replace the right hand side in Equation (27) with the term $\max_{T}\{X_1(t)\} + \max_{T}\{X_2(t)\} + ... + \max_{T}\{X_n(t)\}$. It is of course still unlikely (but less) than $n-1$ loads will attain their maximum at the same time but if the argumentation still hold in the sense that the probability of simultaneous occurrence of a maximum of two of the loads is negligible then Equation (54) may be solved by evaluating the maximum load for the individual loads for the given reference period and combining them in accordance with the scheme shown in Equation (55)

$$
\begin{aligned}
Z_1 &= \max_T\{X_1(t)\} + X_2(t^*) + X_3(t^*) + \ldots + X_n(t^*) \\
Z_2 &= X_1(t^*) + \max_T\{X_2(t)\} + X_3(t^*) + \ldots + X_n(t^*) \\
&\vdots \\
Z_n &= X_1(t^*) + X_2(t^*) + X_3(t^*) + \ldots + \max_T\{X_n(t)\}
\end{aligned}
\tag{55}
$$

and approximating the maximum combined load $X_{\max}(T)$ by

$$X_{\max}(T) \approx \max_i\{Z_i\} \tag{56}$$

This approximation is called Turkstra's rule and is commonly used as a basis for codified load combination rules as will be seen later.

The Ferry Borges – Castanheta load combination rule

A more refined approximation to the load combination problem is based on the load model due to Ferry Borges and Castanheta. This load model builds on a highly simplified representation of the real load processes but facilitates a solution of the load combination problem as defined by equation (54) by use of modern reliability methods such as FORM and SORM described in a later section.

It is assumed that new realizations of each of the individual loads X_i take place at equidistant intervals in time τ_i and are constant in between. This is illustrated in Figure 4 where the reference period T has been divided into n_i intervals of equal length $\tau_i = T / n_i$. n_i is called the repetition number for the i^{th} load. It is assumed that the loads in the n_i time intervals are identically distributed and mutually independent random variables with a point in time probability distribution function $F_{X_i}(x_i)$. This type of load modeling corresponds to a rectangular wave pulse process. The n_i pulses of the process may simply be interpreted as a vector of mutually independent random variables.

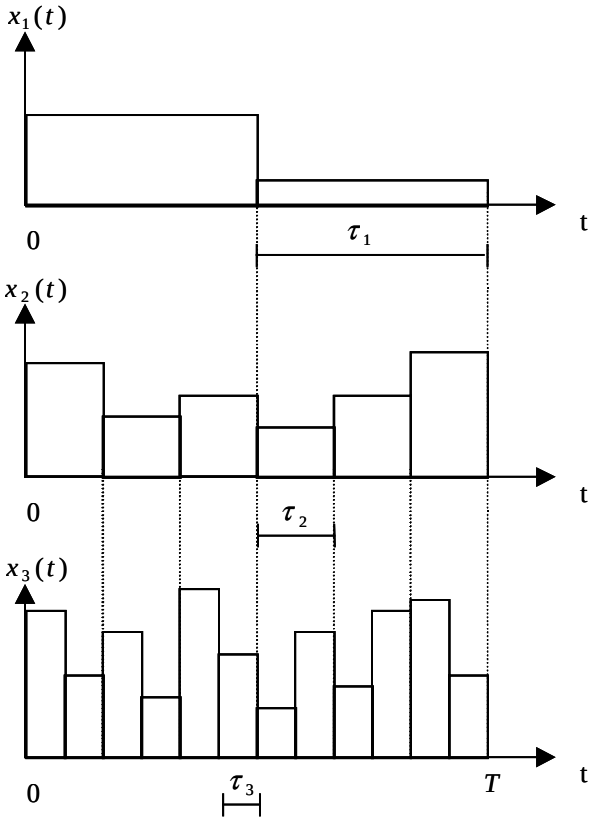


Figure 13 Illustration of the Ferry Borges - Castanheta load process.

The distribution function for the maximum value of the load X_i in the reference period T is then given by

$$F_{\max\limits_{T} X_i}(x_i) = (F_{X_i}(x_i))^{n_i} \tag{57}$$

When combining the load processes $X_1, X_2, .., X_r$, it is furthermore assumed that these are mutually independent with integer repetition numbers n_i

$$n_1 \le n_2 \le ..n_i \le .. \le n_r \tag{58}$$

such that

$$\frac{n_i}{n_{i-1}} \in Z_+ \text{ for } i \in \{2,3,..,r\} \tag{59}$$

where N_+ is the set of real positive numbers.

The combined load effect following the combination rule due to Ferry Borges – Castanheta may now be established by evaluation of

$$X_{\max}(T) \approx \max_i \{Z_i\} \tag{60}$$

where Z_i, $i = 1,2,..,2^{r-1}$ correspond to different cases of load combinations. For $r = 3$ the load cases to be considered are given in Table 5.

Load combination	Repetition numbers		
1	n_1	n_2 / n_1	n_3 / n_2
2	1	n_2	n_3 / n_2
3	n_1	1	n_3 / n_1
4	1	1	n_3

Table 5 Load combinations and repetition number to be considered for the Ferry Borges – Castanhata load combination rule.

3.5 Modeling of resistances

In the following resistance variables are understood as any random variable effecting the ability of a considered technical component or system to withstand the loading acting from the out side. The resistance is thus to be understood as a characteristic of the interior of the considered component or system.

As for the loading the following treatment of resistance variables will be related to
structural reliability analysis but as before the philosophy and concepts will be transferable to other fields of risk and reliability engineering.

Typical resistance variables in structural reliability analysis are

- Geometrical uncertainties
- Material characteristics

- Model uncertainties

As for the probabilistic modeling of loads the important aspects for the probabilistic modeling of resistances are to adequately represent their random variations both in time and space.

Having identified the characteristics of the considered resistance the probabilistic modeling may proceed by

- précising the definition of the random variables used to represent the uncertainties in the resistances
- selecting a suitable distribution type to represent the random variable
- to assign the distribution parameters of the selected distribution.

The probabilistic characteristics for the above mentioned types of resistances are in general rather different, however, some common features apply for their probabilistic modeling and these will be discussed in the following. Detailed probabilistic models for a comprehensive list of resistance variables are given in JCSS [8] and [9].

3.5.1 Geometrical uncertainties

Geometrical characteristics relate to the dimensions of the considered component or system. Typical examples are the concrete cover of reinforced concrete structures, out of straightness of steel columns and the eccentricity of the loading of columns.

The most important aspect for the probabilistic modeling of geometrical quantities is their spatial variability. Usually their time variation may be assumed to be of no relevance.

At the time of design the geometry is uncertain and design specifications together with specifications for the control of the execution are the only available means for limiting the uncertainty. On the basis of such specifications it is possible to set up prior probabilistic models for the geometrical characteristics.

As the absolute value of the deviations of geometry relative to the specified values are governed by tolerance specifications the uncertainties of geometrical quantities tend to have a decreasing influence for increasing structural dimensions.
When the structure has been realised the geometry of the structure may be assessed by measurements. Depending on the geometrical characteristic at hand the measurements may be more or less associated with uncertainty them selves but measurements are valuable means of updating the probabilistic model whereby a posterior probabilistic model may be achieved.

3.5.2 Material characteristics

At the time of design the uncertainties associated with material characteristics have origin in the following sources of uncertainty

- Unknown materials manufacturer
- Unknown production batch(es)
- Random variations of the material characteristics within a given batch
- Uncertainty in relating the material characteristic as estimated on test specimens to the same material characteristic in a real structural component
- Random variations in the quality of the execution of the structure

It is important that each of these sources of uncertainties are taken into account when a probabilistic model for a resistance variable is being formulated.

Assuming that a probability density function $f_X(x)$ for the considered material characteristic X may be postulated the uncertainties referred to under the first three bullets may be modeled on the basis of the following so-called mixed probability distribution model

$$f_X(x) = p(M_1) \cdot f_{X|M_1}(x) + p(M_2) \cdot f_{X|M_2}(x) + .. + p(M_n) \cdot f_{X|M_n}(x)$$
$$\sum_{i=1}^{n} p(M_i) = 1 \tag{61}$$

where n is the number of possible manufactures and $p(M_i)$ is the probability that the material is delivered from manufacturer i and

$$f_{X|M_1}(x) = q(B_1) \cdot f_{X|M_1 \cap B_1}(x) + q(B_2) \cdot f_{X|M_1 \cap B_2}(x) + .. + q(B_m) \cdot f_{X|M_1 \cap B_{m1}}(x)$$
$$\sum_{i=1}^{m} q(B_i) = 1 \tag{62}$$

where m is the number of possible batches and $q(B_i)$ is the probability that the material is taken from batch i.

The probabilistic modeling of the material characteristic for given batch and for given manufacturer may in principle be performed as described in the section on model fitting and estimation. The modeling of the uncertainties associated with the use of material characteristics estimated on the basis of test specimens is a model uncertainty. Model uncertainties are treated in a later section.

In the following some guidelines are given to the selection of distribution functions.

The normal distribution

Whenever the random material characteristic at hand X may be formulated as the sum of the mutually independent random variations of other characteristics the probability distribution function for X will tend to be normal distributed as a consequence of the central limit theorem.

As an example the strength of materials, which may be modeled as system of individual components acting in parallel may be appropriately modeled by the normal distribution. Such a material model is often used in the probabilistic modeling g of the strength of e.g. parallel wire cables.

Even though physical arguments in some cases may be given for the inappropriateness of the normal distribution as a model for a particular material characteristic, due to the possibility for negative realizations of the characteristic, the normal distribution function may still be adequate for practical purposes, however, it should be checked that the probability of negative realizations is negligible.

The log-normal distribution

Whenever the random material characteristic at hand X may be formulated as the product of a number of mutually independent random variations the probability distribution function for X tends to be log-normal distributed.
The log-normal distribution has the advantage that it excludes negative realizations for the considered random material characteristic.

The Weibull distribution

Whenever the random material characteristic at hand X is governed by the size of the largest defect or smallest strength within a certain volume the distribution function for the material characteristic tends to be Weibull distributed.
The Weibull distribution has a large practical importance due to its ability to take into account the so-called volume effect exhibited by materials, which strength may be represented by the strength of a series system with a certain number of mutually independent components. Examples hereof are the strength characteristics of concrete, the strength characteristics of wood, the static and fatigue strength of wires and the static and fatigue capacity of welded connections.

3.6 Model uncertainties

Probabilistic models for uncertain load and resistance characteristics may in principle be formulated at any level of approximation within the range of a purely scientific mathematical description of the physical phenomena governing the problem at hand (micro-level) and a purely empirical description based on observations and tests (macro-level).

In engineering analysis the physical modeling is, however, generally performed at an intermediate level sometimes referred to as the meso-level. Reliability analysis will, therefore, in general be based on a physical understanding of the problem but due to various simplifications and approximations it will always to some extent be empirical. This essentially means that if experimental results of e.g. the ultimate capacity of a portal steel frame are compared to predictions obtained through a physical modeling, which is omitting the effect of non-linearity then there will be a lack of fit. The lack of fit introduces a so-called model uncertainty, which is associated with the level of approximation applied in the physical formulation of the problem. It is important that the model uncertainty is fully appreciated and taken into account in the uncertainty modeling.

The uncertainty associated with a particular model may be obtained by comparing experimental results x_{exp} with the values predicted by the model given the experiment conditions x_{mod}. Defining the model uncertainty M as a factor to be multiplied on the value predicted by the applied model X_{mod} in order to achieve the desired uncertain load or resistance characteristic X, i.e.

$$X = M \cdot X_{\text{mod}} \tag{63}$$

the model uncertainty M may be assessed through observations of m where

$$m = \frac{x_{\text{mod}}}{x_{\text{exp}}} \tag{64}$$

Model uncertainties defined in this way have mean value equal to 1 if they are unbiased. Typical standard deviations for good models may be in the order of magnitude of 2-5 % whereas models for e.g. the shear capacity of concrete structural members the standard deviation is in the range of 10–20 %.

When the model uncertainty M is defined as in Equation (63) it is sometimes convenient to model the probability distribution function $f_M(m)$ by a log-normal distribution, whereby if the uncertain load or resistance variable at hand X_{mod} is also modeled log-normal distributed the product i.e. X is also log-normal distributed.

4. Estimation and Model Building - Classical and Bayes

4.1 Introduction

One of the important tasks in risk and reliability analysis is to establish probabilistic models for the further statistical treatment of uncertain load and resistance variables in the reliability assessment.

In the literature a large number of probabilistic models may be found. E.g. in the Probabilistic Model Code by the Joint Committee on Structural Safety [10] where probabilistic models may be found for the description of the strength characteristics of steel and concrete materials and for the description of load and load effects covering most structural engineering application areas. However it is not always the case that an appropriate probabilistic model for the considered problem is available. In practice two situations may thus be distinguished namely, the situation where a new probabilistic model is formulated from the very beginning and the situation where an already existing probabilistic model is updated on the basis of new information.

The formulation of probabilistic models may be based on data (frequentistic information) alone but most often data are not available to the extent where this is possible. However, in such cases it is usually possible to base the model building on physical arguments, experience and judgment (subjective information). If also some data are available the subjective information is combined with the frequentistic information and the resulting probabilistic is of a Bayesian

nature. It should be emphasised that on the one hand the probabilistic model should aim for simplicity and, on the other hand the model should be close enough to allow for including important data information collected during the lifetime of the considered technical system, and thereby updating the probabilistic model of the considered problem. In this way uncertainty models, which initially are based entirely on subjective information will, as new information is collected, eventually be based on objective information.

In essence the model building process consists of two steps, namely the selection of the appropriate distribution family for the considered random variable or process and thereafter the estimation of the parameters of this distribution.

Typically the choice of the distribution function family may be based mainly on subjective information whereas the parameters of the distribution function are estimated on the basis of the available data and experience.

The principle for establishing a probabilistic model is illustrated in Figure 14.

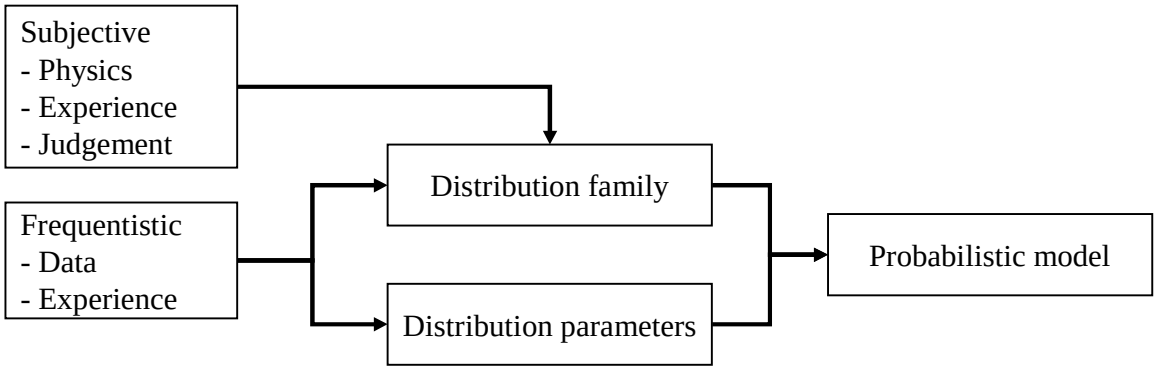


Figure 14 Illustration of the formulation of probabilistic models of uncertain variables.

As the probabilistic models are based on both frequentistic information and subjective information these are Bayesian in nature.

In the following we will consider only the probabilistic modeling of random variables but the described approach applies with some extensions also to the probabilistic modeling of random processes and random fields.

First the issue of choosing an appropriate distribution function family is addressed and thereafter the issue of estimating the parameters of the selected distribution function is considered. As it is often the case in practical engineering problems that a probabilistic model is already available we thereafter finally consider a Bayesian approach to updating of such models based on new data.

4.2 Selection of distribution function

In general the distribution function for a given random variable of stochastic process is not know and must thus be chosen on the basis of frequentistic information, physical arguments or a combination of both.

A formal classical approach for the identification of an appropriate distribution function on the basis of statistical evidence is to

- Postulate a hypothesis for the distribution family
- Estimate the parameters for the selected distribution on the basis of statistical data
- Perform a statistical test to attempt to reject the hypothesis

If it is not possible to reject the hypothesis the selected distribution function is appropriate for the modeling of the considered random variable. If the hypothesis is rejected a new hypothesis must be postulated and the process repeated.

This procedure follows closely the classical frequentistic approach to statistical analysis. However, in practical engineering applications the procedure has limited value. This not least due to the fact that the amount of available data most often is too limited to form the solid basis of a statistical test but also because the available tests are not really good enough in the sense that they may lead to the false conclusions.

In practice it is often the case that physical arguments can be formulated for the choice of distribution function and statistical data are therefore merely used for the purpose of checking whether the anticipated distribution function is plausible.

A practically applicable approach for the selection of the distribution function for the modeling of a random variable is thus

First to consider the physical reasons why the quantity at hand may belong to one or the other distribution family.

Thereafter to check whether the statistical evidence are in gross contradiction with the assumed distribution using e.g. probability paper as explained later.

Finally in the next section it is described how to fit the parameters of the selected distribution taking both into account the available statistical evidence and any prior information at hand.

4.2.1 Use of probability paper

Having selected a distribution family for the probabilistic modeling of a random variable probability paper is an extremely useful tool for the purpose of checking the plausibility of the selected distribution family.

A probability paper for a given distribution family is constructed such that the cumulative probability density function (or the complement) for that distribution family will have the shape of a straight line when plotted on the paper. A probability paper is thus constructed by a non-linear transformation of the y-axis.

For a normal distributed random variable the cumulative probability distribution is given as

$$F_X(x) = \Phi\left(\frac{x-\mu_X}{\sigma_X}\right) \tag{65}$$

where μ_X and σ_X are the mean value and the standard deviation of the normal distributed random variable and where $\Phi(\cdot)$ is the standard normal probability distribution function. By inversion of Equation (65) we get

$$x = \Phi^{-1}(F_X(x)) \cdot \sigma_X + \mu_X \tag{66}$$

Now by plotting x against $\Phi^{-1}(F_X(x))$, see also Figure 15 it is seen that we will obtain a straight line with slope depending on the standard deviation of the random variable X and crossing point with the y-axis depending on the mean value of the random variable. Such a plot is sometimes called a quantile plot.

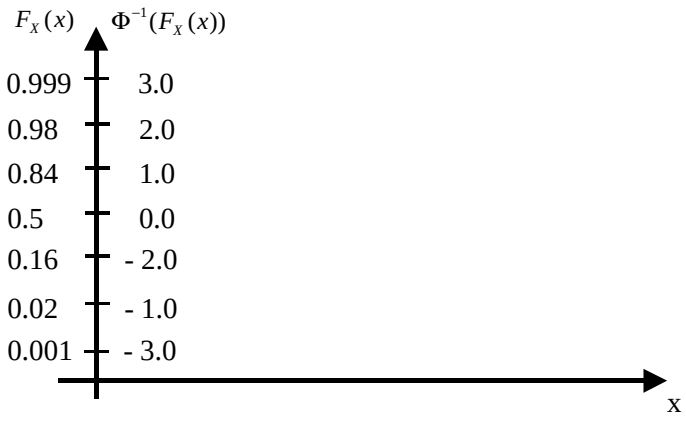


Figure 15 Illustration of the non-linear scaling of the y-axis for a normal distributed random variable.

Also in Figure 15 is given the scale of the non-linear y-axis corresponding to the linear mapping of the observed cumulative probability densities. In probability papers typically only this non-linear scale is given.

Probability papers may also be constructed graphically. In Figure 16 the graphical construction of a normal probability paper is illustrated.

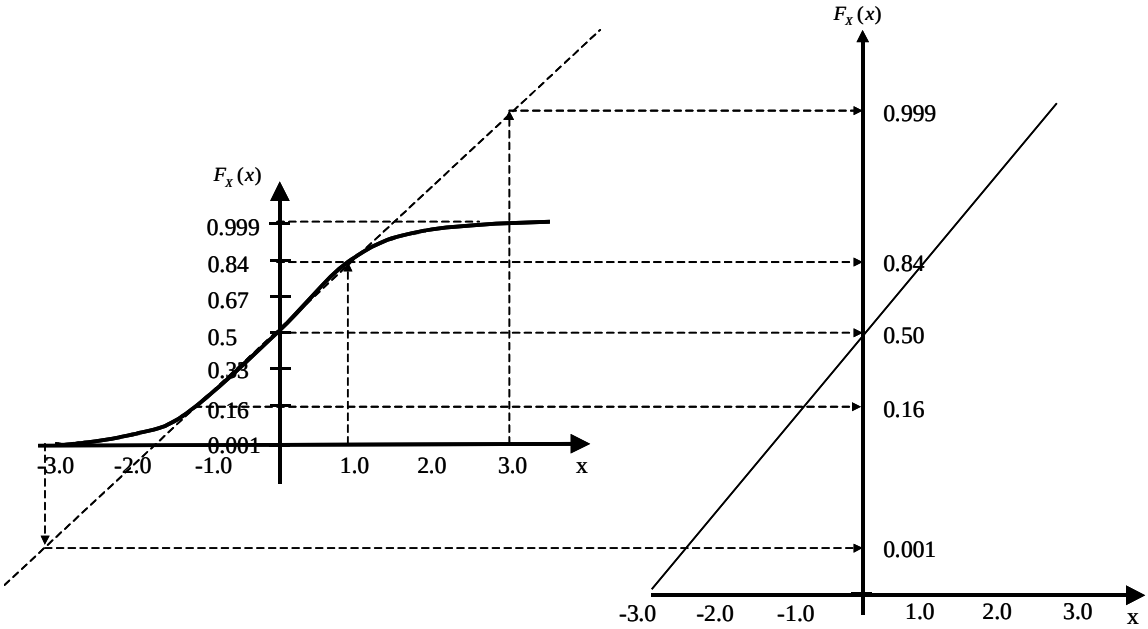


Figure 16 Illustration of the graphical construction of a normal distribution probability paper.

Various types of probability paper are readily available in the literature.

Given an ordered set of observed values $\mathbf{x} = (x_1, x_2, .., x_N)^T$ of a random variable the cumulative distribution function may be evaluated as

$$F_X(x_i) = \frac{i}{N+1} \tag{67}$$

In Table 6 an example is given for a set of observed concrete cube compression strengths together with the cumulative distribution function values as calculated using Equation (67). In Figure 17 the cumulative distribution values are plotted in a normal distribution probability paper.

A first estimate of the distribution parameters may be readily determined from the slope and the position of the best straight line through the plotted cumulative distribution values. In the next chapter the problem of parameters estimation is considered in more detail.

n	x_n	$F_X(x_n)$
1	24.4	0.047619048
2	27.6	0.095238095
3	27.8	0.142857143
4	27.9	0.19047619
5	28.5	0.238095238
6	30.1	0.285714286
7	30.3	0.333333333
8	31.7	0.380952381
9	32.2	0.428571429
10	32.8	0.476190476
11	33.3	0.523809524
12	33.5	0.571428571
13	34.1	0.619047619
14	34.6	0.666666667
15	35.8	0.714285714
16	35.9	0.761904762
17	36.8	0.80952381
18	37.1	0.857142857
19	39.2	0.904761905
20	39.7	0.952380952

Table 6 Ordered set of observed concrete cube compression strengths and the calculated cumulative distribution values.

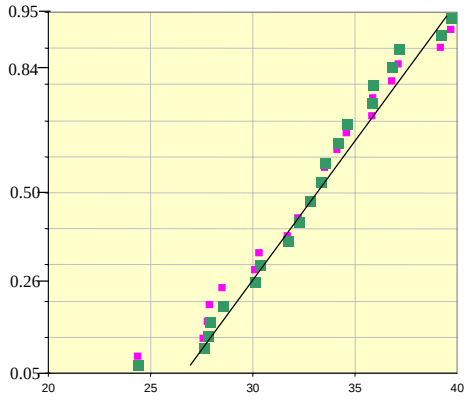


Figure 17 Concrete cube compression strength data plotted in normal distribution paper.

It is seen from Figure 17 that the observed cumulative distribution function does not fit too well on a straight line. Especially in the lower tail area but also in the upper the deviations from the straight line are obvious. This is also to be expected as the distribution family, which based alone on physical arguments for the concrete cube compression strength cannot be a normal distribution, but rather a log-normal distribution.

When using probability paper for the consideration of extreme phenomena such as e.g. the maximum water level in a one year period the probability paper may also be used for the purpose of estimating the values of the water level with a certain return period i.e. for the purpose of extrapolation see e.g. Schneider [3]. However as always when extrapolating, extreme care must be exercised.

4.3 Estimation of distribution parameters

There are in principle two different methods to estimate the distribution parameters on the basis of data, namely the methods of point estimates and the methods of interval estimates. In the following, however, only two of the methods of point estimates will be explained, namely the method of moments and the method of maximum likelihood as these have proven especially useful in practical risk and reliability engineering analysis.

The method of moments

Assuming that the considered random variable X may be modeled by the probability density function $f_X(x;\boldsymbol{\theta})$ where $\boldsymbol{\theta}=(\theta_1,\theta_2,..,\theta_k)^T$ are the distribution parameters, the first k moments $\boldsymbol{\lambda}=(\lambda_1,\lambda_2,..,\lambda_k)$ of the random variable X may be written as

$$\begin{aligned}\lambda_j &= \int_{-\infty}^{\infty} x^j \cdot f_X(x)dx \\ &= \lambda_j(\theta_1,\theta_2,..,\theta_k)\end{aligned} \tag{68}$$

If the random sample, which we wish to estimate the distribution parameters $\boldsymbol{\theta}=(\theta_1,\theta_2,..,\theta_k)^T$ from are collected in the vector $\mathbf{x}=(x_1,x_{2,},..,x_n)^T$ the corresponding k sample moments may be calculated as

$$m_j=\frac{1}{n}\sum_{i=1}^{n}x_i^j \tag{69}$$

By equating the k sample moments to the k equations for the moments for the random variable X a set of k equations with the k unknown distribution parameters are obtained, the solution of which gives the point estimates of the distribution parameters.

The method of maximum likelihood

This method may be somewhat more difficult to use than the method of moments but has a number of very attractive properties, which makes this method especially applicable in engineering risk and reliability analysis.

The principle of the method is that the parameters of the distribution function are fitted such that the probability (likelihood) of the observed random sample is maximized.

Let the random variable of interest X have a probability density function $f_X(x;\boldsymbol{\theta})$ where $\boldsymbol{\theta}=(\theta_1,\theta_2,..,\theta_k)^T$ are the distribution parameters to be fitted.

If the random sample, from which we wish to estimate the distribution parameters $\boldsymbol{\theta}=(\theta_1,\theta_2,..,\theta_k)^T$ are collected in the vector $\mathbf{x}=(x_1,x_{2,},..,x_n)^T$ the likelihood $L(\boldsymbol{\theta}|\mathbf{x})$ of the observed random sample is defined as

$$L(\boldsymbol{\theta}|\mathbf{x})=\prod_{i=1}^{n}f_X(x_i|\boldsymbol{\theta}) \tag{70}$$

The maximum likelihood point estimates of the parameters $\boldsymbol{\theta}=(\theta_1,\theta_2,..,\theta_k)^T$ may now be obtained by solving the following optimization problem

$$\min_{\boldsymbol{\theta}}(-L(\boldsymbol{\theta}|\mathbf{x})) \tag{71}$$

Instead of the likelihood function it is advantageous to consider the log-likelihood $l(\boldsymbol{\theta}|\mathbf{x})$ i.e.

$$l(\boldsymbol{\theta}|\mathbf{x})=\sum_{i=1}^{n}\log(f_X(x_i|\boldsymbol{\theta})) \tag{72}$$

One of the most attractive properties of the maximum likelihood method is that when the number of samples i.e. n is sufficiently large the distribution of the parameter estimates converges towards a normal distribution with mean values $\boldsymbol{\mu}_{\Theta\Theta}$ equal to the point estimates, i.e.

$$\boldsymbol{\mu}_{\Theta\Theta} = (\theta_1^*, \theta_2^*, .., \theta_n^*)^T \tag{73}$$

The covariance matrix $\mathbf{C}_{\Theta\Theta}$ for the point estimates may readily be obtained by

$$\mathbf{C}_{\Theta\Theta} = \mathbf{H}^{-1} \tag{74}$$

where $\mathbf{H}$ is the Fischer information matrix with components determined by the second order partial derivatives of the -log-likelihood function taken in the optimum, i.e.

$$H_{ij} = \frac{\partial^2 - l(\boldsymbol{\theta}|\mathbf{x})}{\partial\theta_i \partial\theta_j}\Big|_{\boldsymbol{\theta}=\boldsymbol{\theta}^*} \tag{75}$$

Example

Consider again the experimental results of the concrete cube compression strengths given in Table 6. Assuming that distribution function for the concrete cube compression strength is normal distributed we want to estimate the parameters on the basis of the experiment results.

It can be shown that the equations for the moments of a normal distribution in terms of the distribution parameters are given as

$$\begin{aligned} \lambda_1 &= \mu \\ \lambda_2 &= \mu^2 + \sigma^2 \end{aligned} \tag{76}$$

Analyzing the sample data the first two sample moments are found as

$$\begin{aligned} m_1 &= 32.67 \\ m_2 &= 1083.36 \end{aligned} \tag{77}$$

The point estimates of the parameters μ, σ may now be determined by solving the equations

$$\begin{aligned} \mu &= 32.67 \\ \mu^2 + \sigma^2 &= 1083.36 \end{aligned} \tag{78}$$

giving

$$\begin{aligned} \mu &= 32.67 \\ \sigma &= 4.05 \end{aligned} \tag{79}$$

Using the method of maximum likelihood the maximum likelihood function is readily written as

$$L(\theta|x) = \left(\frac{1}{2\pi\theta_1}\right)^n \cdot \exp\left(-\frac{1}{2}\sum_{i=1}^{n}\frac{(x_i - \theta_2)^2}{\theta_1^2}\right) \tag{80}$$

and correspondingly the log-likelihood function as

$$l(\boldsymbol{\theta}|\mathbf{x}) = n \cdot \ln\left(\frac{1}{\sqrt{2\pi}\theta_1}\right) - \frac{1}{2}\sum_{i=1}^{n}\frac{(x_i-\theta_2)^2}{\theta^2} \tag{81}$$

The mean values of the estimates may be determined by solving the following equations

$$\begin{aligned}\frac{\partial l}{\partial \theta_1} &= -\frac{n}{\theta_1} + \frac{1}{\theta_1^3}\sum_{i=1}^{n}(x_i-\theta_2)^2 = 0\\ \frac{\partial l}{\partial \theta_2} &= -\frac{1}{\theta_1^2}\sum_{i=1}^{n}(x_i-\theta_2) = 0\end{aligned} \tag{82}$$

yielding

$$\begin{aligned}\theta_1 &= \sqrt{\frac{\sum_{i=1}^{n}(x_i-\theta_2)^2}{n}}\\ \theta_2 &= \frac{1}{n}\sum_{i=1}^{n}x_i\end{aligned} \tag{83}$$

which by using the sample data gives

$$\begin{aligned}\theta_1 &= \sigma = 4{,}04\\ \theta_2 &= \mu = 32{,}665\end{aligned}$$

Not surprisingly the same result as the method of moments.

As mentioned previously the covariance matrix $\mathbf{C}_{\Theta\Theta}$ for the parameters estimates may be determined through the information matrix $\mathbf{H}$ containing the second-order partial derivatives of the log-likelihood function, see Equation (75). The information matrix may be found to be

$$H = \begin{pmatrix} -\frac{2n}{\theta_1} + \frac{3\sum_{i=1}^{n}(x_i-\theta_2)^2}{\theta_1^4} & -\frac{2\sum_{i=1}^{n}(x_i-\theta_2)}{\theta_1^3} \\ -\frac{2\sum_{i=1}^{n}(x_i-\theta_2)}{\theta_1^3} & -\frac{n}{\theta_1^2} \end{pmatrix} \tag{84}$$

whereby the covariance matrix is evaluated using the sample data as

$$C_{\Theta\Theta} = H^{-1} = \begin{pmatrix} 0{,}836 & 0 \\ 0 & 0{,}1647 \end{pmatrix} \tag{85}$$

In a probabilistic modeling where the concrete cube compression strength enters as a random variable it is then possible to take into account the statistical uncertainty associated with the estimates of the distribution parameters for the distribution function simply by including the distribution parameters in the reliability analysis as normal distributed variables with the evaluated mean values and covariances.

4.4 Bayesian estimation methods

A typical situation in risk and reliability analysis is that a prior probabilistic model for a random variable is available, e.g. a distribution function and distribution parameters have been chosen and estimated based on previous experiment results, experience and professional judgment. Then additional data becomes available and it is desired to update the distribution parameters of the prior probabilistic model on the basis of these data, weighing the confidence in the prior model consistently with the evidence provided by the new data.

Consider a random variable X with density function $f_X(x)$. If θ denotes a vector with parameters defining the distribution for X the density function of the random variable X can be written

$$f_X(x,\theta) \tag{86}$$

If X is normal distributed then θ could contain the mean and the standard deviation of Z.

If the parameters θ are uncertain then $f_X(x,\theta)$ can be considered as a conditional density function: $f_X(x|\Theta)$. θ denotes a realizations of Θ .The prior (initial) density function for the parameters Θ is denoted $f'_\Theta(\theta)$ and is denoted the prior density function.

It is assumed that n observations (realizations) of the random variable X are available making up a sample $\mathbf{x}=(x_1,x_2,...,x_n)$ of size n. The realizations are assumed to be independent. The updated density function $f''_\Theta(\theta|\mathbf{x})$ [I61] of the uncertain parameters Θ given the realizations is denoted the posterior density function and is given by, see JCSS [11].

$$f''_\Theta(\theta|\mathbf{x})=\frac{f_N(\mathbf{x}|\theta)\,f'_\Theta(\theta)}{\int f_N(\mathbf{x}|\theta)\,f'_\Theta(\theta)d\theta} \tag{87}$$

where $f_N(\mathbf{x}\theta)=\prod_{i=1}^{n} f_X(x_i|\theta)$ is the probability density at the given observations assuming that the distribution parameters are θ. The integration in Equation (87) is over all possible values of θ.

The updated density function of the random variable X given the realization $\mathbf{x}$ is denoted the predictive density function and is defined by,

$$f_X(x|\mathbf{x})=\int f_X(x|\theta)\,f''_\Theta(\theta|\mathbf{x})d\theta \tag{88}$$

Given the distribution function for the random variable X, the prior distribution is often chosen such that the posterior distribution will be of the same type as the prior distribution (a so-called conjugated prior). In the literature a number of prior, posterior and predictive distribution functions can be found, see e.g. JCSS [11]. Analytical solutions concerned with the following problems can be found for

- Normal distribution with unknown mean
- Normal distribution with unknown standard deviation
- Normal distribution with unknown mean and standard deviation
- Gumbel distribution
- Weibull distribution
- Exponential distribution
- Bernoulli distribution
- Poisson distribution
- Multidimensional Normal distribution with unknown means
- Multidimensional Normal distribution with unknown standard deviations

- Multidimensional Normal distribution with unknown means and standard deviations

The parameters in the prior distribution can be chosen or calculated in such a way that the prior reflects

known (initial) observations of the random variables $\mathbf{X}$ from which estimates of the parameters in the prior distribution can be calculated.
subjective knowledge on the distribution of the parameters Θ in the distribution of $\mathbf{X}$.

In this way it is possible to choose a prior distribution, which reflects a range of situations from very good prior knowledge on the parameter (small standard deviation) to almost no knowledge on the parameters (large standard deviation).

Example

As an example on updating of random variables consider the probabilistic modeling of the yield stress of the steel bar. The prior probabilistic model for the yield stress of the steel bar was assumed to normal distributed with known (deterministic) standard deviation σ_{f_y} equal to 17.5 MPa and uncertain mean value. The mean value μ_{f_y} was assumed normal distributed with known mean value $\mu' = 350\text{MPa}$ and standard deviation $\sigma' = 10\text{MPa}$.
Assume now, that 5 tests of the yield stress are performed on steel samples taken from a batch of the same steel material. The test results are $\hat{f}_y = (365,347,354,362,348)$.
Based on the test results the prior probabilistic model for the mean value of the yield stress can be updated using natural conjugate distributions as mentioned earlier.
In the case considered with a normally distributed variable with uncertain mean and known standard deviation the posterior as given in Equation (87) may be found to reduce to (JCSS [11])

$$\varphi_{\mu_{f_y}}(\mu_{f_y}) = \frac{1}{\sqrt{2\pi}\sigma''}\exp\left(-\frac{1}{2}\left(\frac{\mu_{f_y}-\mu''}{\sigma''}\right)^2\right) \tag{89}$$

where

$$\mu'' = \frac{\frac{\mu'}{n}+\frac{\bar{x}}{n'}}{\frac{1}{n'}+\frac{1}{n}} \tag{90}$$

and

$$\sigma'' = \sqrt{\frac{\frac{\sigma_{f_y}^2}{n'}\cdot\frac{\sigma'^2}{n}}{\frac{\sigma'^2}{n'}+\frac{\sigma'^2}{n}}} \tag{91}$$

and

$$n' = \frac{\sigma_{f_y}^2}{\sigma'^2} \tag{92}$$

$\bar{x}$ is the sample mean of the observations, n' is the sample size assumed for the prior distribution of μ_R and n is the sample size for the new sample. In the present example $n' = 3.06$.
Based on the new observations the posterior parameters are $\mu'' = 353.22$ and $\sigma'' = 6.16$. In Figure 18 plots are shown for the prior and the posterior probability density functions for μ_{σ_y}

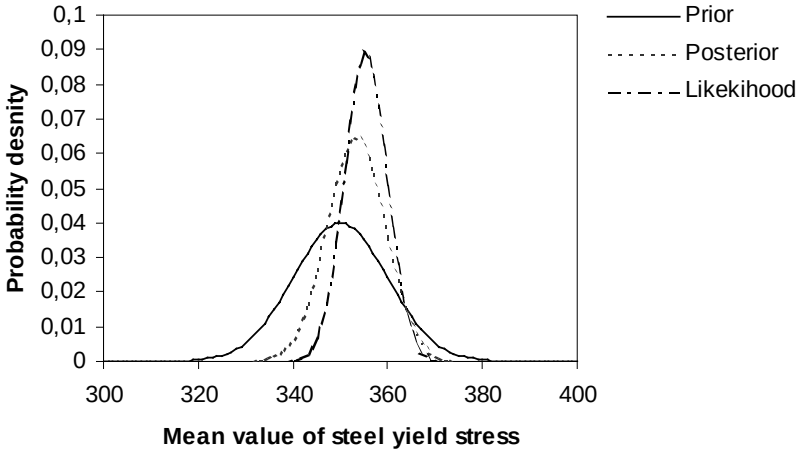


Figure 18 Illustration of prior and posterior probability density functions for the mean value of the steel yield stress. Also the likelihood for the test results are shown.

The likelihood of the observation can be established as

$$L(\mu_{f_y} \left| \hat{f}_y \right.) \propto \prod_1^5 \frac{1}{\sqrt{2\pi}\sigma'} \exp(-\frac{1}{2}\frac{(\hat{f}_{yi} - \mu_{f_y})^2}{\sigma'^2}) \tag{93}$$

The likelihood function is also shown in Figure 18.

It is seen from Figure 18 that the effect of the test results is quite significant. The predictive probability density function for the steel yield stress may according to e.g. JCSS [11] be determined as

$$f_{f_y}\left(f_y \left| \hat{f}_y \right.\right) = \frac{1}{\sqrt{2\pi}\sigma'''} \exp\left(-\frac{1}{2}\left(\frac{f_y - \mu''}{\sigma'''} \right)^2 \right) \tag{94}$$

where

$$\sigma'''^2 = \sigma''^2 + \sigma_{f_y}^2 \tag{95}$$

In Figure 19 the predictive probability distribution and the probability distribution function for the steel yield stress based on the prior information of the mean value are shown.

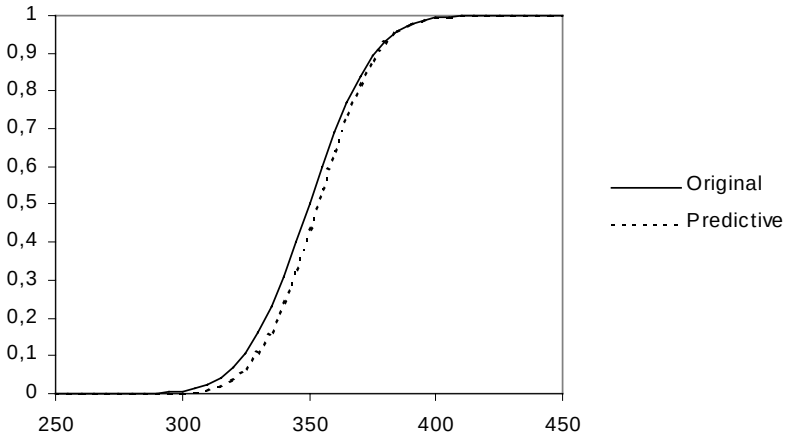


Figure 19 Illustration of original and predictive probability distribution function for the steel yield stress.

The 5% percentile value, which is a typical characteristic value for the steel yield stress is changed from 317 MPa to 322 MPa as a result of the test results.

5 Reliability of Technical Components

5.1 Introduction

Reliability analysis of technical components became a central issue during the Second World War where significant problems were encountered especially in regard to the performance of electrical systems. As an example the war systems of the, at that time modern hangar ships were reported non-operable in up to about 40 % of the time. This situation which could be quite critical in times of war was caused predominantly by failures of electrical components (radio bulbs, etc.) and the efforts initiated at that time in order to improve the performance of the electrical systems may be seen as an initiation point for the analysis of the reliability of technical components.

Since then reliability analysis of technical components has been further developed and adapted for application in a wide range of different industries including the aeronautical industry, the nuclear industry, the chemical industry, the building industry and the process industry. It is important to appreciate that reliability analysis is only one of the constituents of a decision analysis or more popularly speaking a risk analysis, namely the part which is concerned about the quantification of the probability that a considered components or system is in a state associated with adverse consequences, e.g. a state of failure, a state of damage or partial function, etc. The theoretical basis for reliability analysis is thus the theory of probability and statistics and derived disciplines such as operations research, systems engineering and quality control.

Classical reliability theory was, as previously indicated developed for systems consisting of a large number of components of the same type under the same loading and for all practical matters behaving statistically independent. The probability of failure of such components can be interpreted in terms of relative failure frequencies observed from operation experience. Furthermore, due to the fact that failure of the considered type of components develops as a direct consequence of an accumulating deterioration process the main focus was directed towards the formulation of probabilistic models for the estimation of the statistical characteristics of the time until component failure. Having formulated these models the observed relative failure frequencies can be applied as basis for their calibration.

In structural reliability analysis the situation is fundamentally different due to the fact that structural failures are very rare and tend to occur as a consequence of an extreme event such as e.g. an extreme loading exceeding the load carrying capacity i.e. the resistance, which possibly is reduced due to deterioration such as e.g. corrosion or fatigue. In addition to this no useful information can be collected in regard to relative failure frequencies as almost all structural components and systems are unique either due to differences in the choice of material and geometry or by differences in the loading and exposure characteristics.

When considering the estimation of failure probabilities for structural components it is thus necessary to establish a probabilistic modelling of both the resistances and the loads and to estimate the probability of failure on the basis of these. In this process due account must then be given to the inclusion of all available statistical information concerning the material properties and the load characteristics.

In the following sections we shall give an overview first of the classical reliability theory and thereafter consider the problem of structural reliability analysis with a view to the special characteristics of this problem considerations to its solution. In a later chapter we shall then see how the problem may be solved in a practical manner.

5.2 Classical Reliability Analysis

As mentioned previously the classical reliability analysis was developed to estimate the statistical characteristics of the lives of technical components in order to be able to predict the characteristics which are important for the design and operation of systems build up by such components. These characteristics include the expected failure rate, the expected life and the mean time between failures.

Modeling the considered system by means of logical trees where the individual components are represented by the nodes it is possible the assess the key characteristics regarding the performance including e.g. the probability that a

system will fail during a specified period, the positive effect of introducing redundancy into the system and its cost consequences and the effect of inspections and maintenance activities.

The probability of failure of a component is expressed by means of the reliability function $R_T(t)$ defined by

$$R_T(t) = 1 - F_T(t) = 1 - P(T \le t) \tag{96}$$

where T is a random variable describing the time till failure and $F_T(t)$ is its probability distribution function. If the probability density function for T, i.e. $f_T(t)$ is known the reliability function maybe defined alternatively by

$$R_T(t) = 1 - \int_0^t f_T(\tau) d\tau = \int_t^\infty f_T(\tau) d\tau \tag{97}$$

The reliability function thus depends on the type of probability distribution function for the time till failure. In the same way as when considering the probabilistic modeling of load and resistance variables prior information may be utilized when selecting the distribution type for the modeling of the random time till failure for a technical component. The appropriate choice of distribution function then depends on the physical characteristics of the deterioration process causing the failure of the component.

In the literature several models for the time till failure have been derived on the basis of the characteristics of different deterioration processes. These include the exponential distribution, the Weibull distribution, and the Birnbaum and Saunders distribution. In case of a Weibull distribution the reliability function has the following form

$$R_T(t) = 1 - F_T(t) = 1 - (1 - \exp\left[-(\frac{t}{k})^\beta\right]) = \exp\left[-(\frac{t}{k})^\beta\right], \qquad t \ge 0 \tag{98}$$

Having defined the reliability function $R_T(t)$ the expected life may be derived as

$$E[T] = \int_0^\infty \tau \cdot f_T(\tau) d\tau = \int_0^\infty R_T(t) dt \tag{99}$$

which may be seen by performing the integrations in parts

$$E[T] = \int_0^\infty R_T(t) dt = \left[t \cdot R_T(t)\right]_0^\infty + \int_0^\infty t \cdot f_T(t) dt \tag{100}$$

provided that $\lim_{t \to \infty} \left[t \cdot R_T(t)\right]_0^\infty = 0$.

The failure rate is a measure of how the probability of failure changes as function of time. The failure rate thus depends on the reliability function $R_T(t)$. The probability of failure within any given interval $[t, t + \delta\, t]$ is the probability that the actual life lies in the interval and is thus given as

$$P(t < T \le t + \delta\, t) = F_T(t + \delta\, t) - F_T(t) = R_T(t) - R_T(t + \delta\, t) \tag{101}$$

The failure rate function $z(t)$ being the average rate at which failures occur in a given time interval given that the considered component has not failed prior to the interval is thus

$$z(t) = \frac{R_T(t) - R_T(t + \delta\, t)}{\delta\ t\, R_T(t)} \tag{102}$$

The failure rate function for most technical systems is known as the bath-tub curve illustrated in Figure 20.

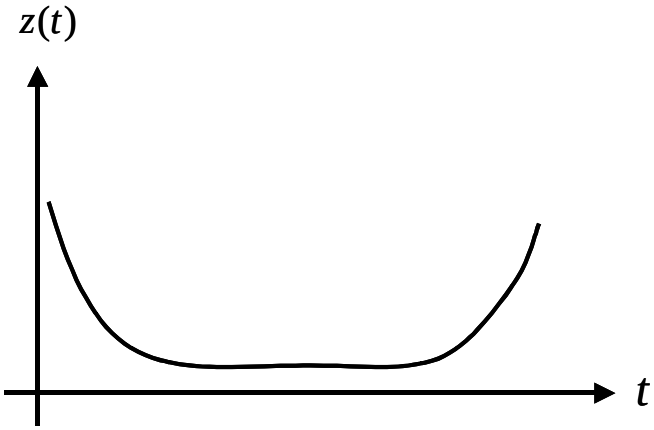


Figure 20 The failure rate function – bath-tub curve.

The bath-tub curve is typical for many technical components where in the initial stages of the life the birth defects, production errors etc are a significant source of failure. When the component has survived a certain time it implies that birth defects were not present and the reliability increases. Thereafter stage of steady state occurs and subsequently a stage of ageing. The steepness of the ageing part of the failure rate function is important. The more pronounced and the steeper the transition is from the steady state stage to the ageing stage of the life of the component the more obvious is the decision on when to exchange or maintain the component.
The shape of the failure rate function has also implications on the meaningful inspection strategies, which may be implemented as a means for condition control of a component. For components exhibiting a constant failure rate function, i.e. components with an exponential distribution as given in Equation (103) for the time till failure, inspections are of little use.

$$f_T(t) = z \cdot \exp(-z \cdot t)) \tag{103}$$

The component does not exhibit any degradation and there is not really anything to inspect. However, for components with a slowly increasing failure rate function inspections may be useful and can be planned such that the failure rate does not exceed a certain critical level. If the failure rate function is at first quasi constant and then followed by an abrupt increase inspections are also of little use, however, in this case a replacement strategy is more appropriate.

The hazard function $h(t)$ is defined as the instantaneous failure rate as the considered interval approaches zero. Thus the hazard function is given as

$$h(t) = \lim_{\delta\ t \to 0} \frac{R_T(t) - R_T(t+\delta\ \mathrm{t})}{\delta\ t\ R_T(t)} = \frac{1}{R_T(t)}\left[-\frac{d}{dt} R_T(t)\right] = \frac{f_T(t)}{R_T(t)} \tag{104}$$

and the probability that a component having survived up till the time t will fail in the next small interval of time dt is then h(t) dt.

An important issue is the assessment of failure rates on the basis of observations. As mentioned previously data on observed failure rates may be obtained from databanks of failures from different application areas. Failure rates may be assessed on the basis of such data by

$$z(t) = \frac{n_f}{\tau \cdot n_i} \tag{105}$$

where n_f is the number of observed failure in the time interval τ and n_i is the number of components at the start of the considered time interval. Care must be exercised when evaluating failure rates on this basis. If the components are not new in the beginning of the considered time interval the failure rates may be over estimated and if the interval is too short no observed failures may be present. For such cases different approaches to circumvent this problem may be found in the literature, see e.g. Stewart and Melchers [2]. Alternatively the failure rates may also be assessed by means

of e.g. Maximum-likelihood estimation where the parameters of the selected probability distribution function for the time till failure are estimated on the basis of observed times till failures.

Due to the lack of data and general uncertainties associated with the applicability of the available data for a specific considered case failure rates may themselves be modeled as uncertain. The basis for the a-priori assessment of the uncertainty associated with the failure rates may be established subjectively or preferably as a bi-product of the Maximum-Likelihood estimation of the distribution parameters of the probability distribution function for the time till failures. Having established an a-priori model for the failure rate for a considered type of component another important issue is how to update this estimate when new or more relevant information about observed failures become available.

Applying the rule of Bayes the a-posteriori probability density function for the failure rate may be established as

$$f_Z^{''}(z) = \frac{L(E|z) \cdot f_Z^{'}(z)}{\int_0^{\infty} L(E|z) \cdot f_Z^{'}(z) dz} \tag{106}$$

Assuming that the time till failure for a considered component is exponential distributed and if the additional information consists of a number of observed failures, for a given reference period the likelihood function may be estimated by means of the Poisson distribution

$$L(E|z) = \frac{\exp(-z \cdot \tau)(-z \cdot \tau)^n}{n!} \tag{107}$$

Table 7 - 10 include a number of failure rates, adapted from Steward and Melchers [2] for various types of components in the mechanical, electrical and offshore industry.

Environmental Stress	*Modifier for failure rate*
High temperature	x 1.75
High radiation	x 1.25
High humidity	x 1.5
High vibration	x 2.0

Table 7 Environmental factors for temperature instrument, control and sensor reliability data, Steward and Melchers [2] (Source. Adapted from IEEE (1984))

	Failures/10^6 hours			*Failures/10^6 cycles*			*Repair time (hours)*		
Failure Mode	*Low*	*Rec*	*High*	*Low*	*Rec*	*High*	*Low*	*Rec*	*High*
All Modes	0.31	1.71	21.94	0.11	0.75	1.51	0.3	0.74	1.3
Catastrophic	0.13	0.7	9						
Zero or maximum output	0.06	0.31	4.05						
No change of output with change of input	0.01	0.04	0.45						
Functioned without signal	0.03	0.18	2.34						
No function with signal	0.03	0.17	2.16						
Degraded	0.14	0.75	9.65						
Erratic output	0.03	0.17	2.22						
High output	0.03	0.15	1.93						
Low output	0.01	0.06	0.77						
Functioned at improper signal level	0.05	0.29	3.67						
Intermittent operation	0.02	0.08	1.06						
Incipient	0.04	0.26	3.29						

Note: Rec refers to the 'Best estimate'; Low, High refers to the best and worst data points (i.e. this establishes the range)

Table 8 Reliability data for temperature instruments, controls and sensors, Steward and Melchers [2] (Source. Adapted from IEEE (1984))

Population	*Samples*	*Aggregated time in service (10^6 hrs)*		*Number of demands*
17	*10*	*Calendar time*	*Operational time*	
		0.3826	*0.0002*	*1135*

		Failure rate (per 10^6 hrs)			*Repair (manhours9*		
Failure mode	*No. Of Failures*	*Lower*	*Mean*	*Upper*	*Min.*	*Mean*	*Max.*
Critical	80*	120	210	310	-	86	-
	13ç	26000	47000	78000			
Failed to start	75*	100	190	90	24	86	120
	9ç	6200	32000	69000			
Failed while running	5*	2	23	51	3	93	130
	4ç	4600	15000	36000			
Degraded	24*	30	71	120	-	180	-
	3ç	0	14000	45000			
High temperature	22*	22	66	120	6	190	400
	3ç	0	14000	44000			
Low output	1*	0.14	2.6	12	-	-	-
Unknown	1*	0.14	2.6	12	-	96	-
Incipient							
Unknown							
All Modes	303*	680	840	1000	-	81	-
	45ç	87000	180000	280000			

*Note: *denotes calendar time, ç denotes operational time*

Table 9 Reliability data for fire water pumps on offshore platforms, Steward and Melchers [2] (Source. Adapted from OREDA (1984))

Component and Failure mode	Best estimate'	Upper and lower bounds
Electric Motors:		
Failure to start	3e-4/D	1e-4 - 1e-3
Failure to run (normal)	1e-5/hr	3e-6 - 3e-5
Failure to run (extreme environment)	1e-3/hr	1e-4 - 1e-2
Battery Power systems:		
Failure to provide proper output	3e-6/hr	1e-6 - 1e-5
Switches:		
Limit - failure to operate	3e-4/D	1e-4 - 1e-3
Torque - failure to operate	1e-4/D	3e-5 - 3e-4
Pressure - failure to operate	1e-4/D	3e-5 - 3e-5
Manual - fail to transfer	1e-5/D	3e-6 - 3e-5
Contacts short	1e-7/hr	1e-8 - 1e-6
Pumps:		
Failure to start	1e-3/D	3e-4 - 3e-3
Failure to run (normal)	3e-5/hr	3e-6 - 3e-4
Failure to run (extreme environment)	1e-3/hr	1e-9 - 1e-7
Valves (motor operated):		
Fails to operate	1e-3/D	3e-4 - 3e-3
Failure to remain open	1e-4/D	3e-5 - 3e-4
External leak or rupture	1e-8/hr	1e-9 - 1e-7
Circuit breakers:		
Failure to operate	1e-3/D	3e-4 - 3e-3
Premature transfer	1e-6/hr	3e-7 - 3e-6
Fuses		

	Premature, open	1e-6/hr	3e-7 - 3e-6
	Failure to open	1e-5/D	3e-6 - 3e-5
Pipes:			
	< 75mm, rupture	1e-9/hr	3e-11 - 3e-8
	> 75mm, rupture	1e-10/hr	3e-12 - 3e-9
Welds:			
	Leak, containment quality	3e-9/hr	1e-10 - 1e-7

Table 10 Reliability data for mechanical and electrical components, Steward and Melchers [2] (Source. Adapted from IRSS (1975))

5.3 Structural Reliability Analysis

Concerning the reliability of structural components and systems the situation is different in comparison to that of e.g. electrical components. For structural components and systems first of all no relevant failure data are available, secondly failures occur significantly more rare and thirdly the mechanism behind failures is different. Structural failure occur not predominantly due to ageing processes but moreover due to the effect of extreme events, such as e.g. extreme winds, avalanches, snow fall, earth-quakes, or combinations hereof.
For the reliability assessment it is therefore necessary to consider the influences acting from the outside i.e. loads and influences acting from the inside i.e. resistances individually. It is thus necessary to establish probabilistic models for loads and resistances including all available information about the statistical characteristics of the parameters influencing these. Such information is e.g. data regarding the annual extreme wind speeds, experiment results of concrete compression strength, etc. These aspects have been treated in a previous chapter. Due to the fact that a significant part of the uncertainties influencing the probabilistic modeling of loads and resistances are due to lack of knowledge the failure probabilities, which may be assessed on this basis must be understood as nominal probabilities, i.e. not reflecting the true probability of failure for the considered structure but rather reflecting the lack of knowledge available about the performance of the structure.

For a structural component for which the uncertain resistance R may be modeled by a random variable with probability density function $f_R(r)$ subjected to the load s the probability of failure may be determined by

$$P_f = P(R \le s) = F_R(s) = P(R/s \le 0) \tag{108}$$

In case also the load is uncertain and modeled by the random variable S with probability density function $f_S(s)$ the probability of failure is

$$P_f = P(R \le S) = P(R - S \le 0) = \int_{-\infty}^{\infty} F_R(x) f_S(x) dx \tag{109}$$

assuming that the load and the resistance variables are statistically independent. This case is called the fundamental case. The integration in Equation (109) is illustrated in Figure 21.

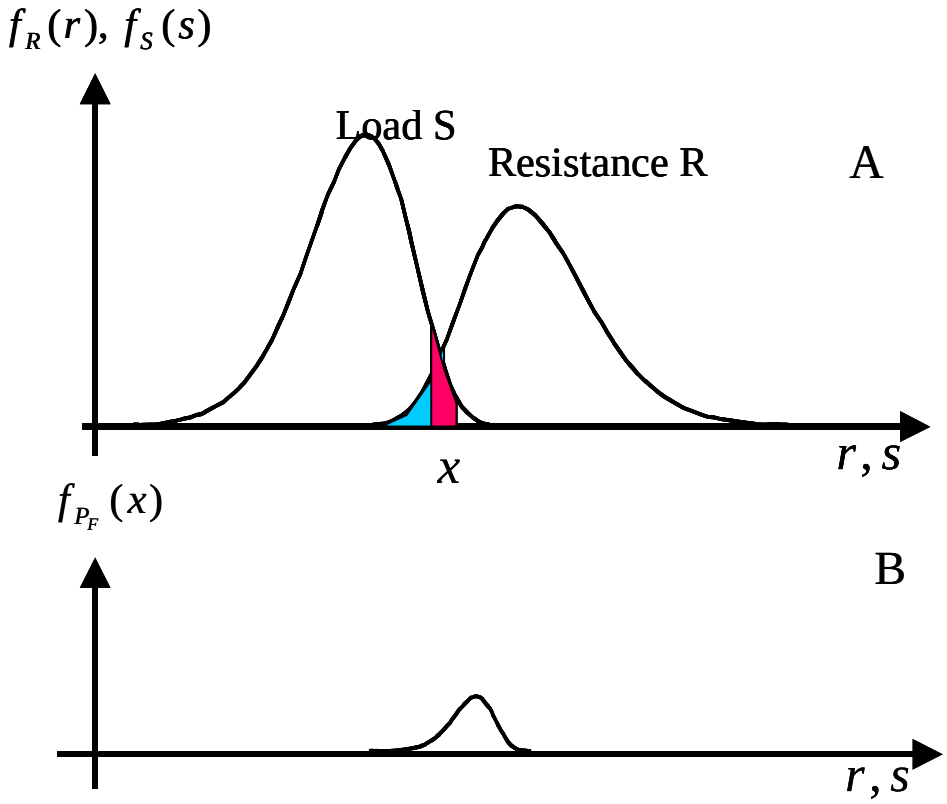


Figure 21 A) Illustration of the integration in Equation (109) and B) the distribution of the failure probability over the realizations of the resistance R and the loading S.

In Figure 21 A) the contributions to the probability integral in Equation (109) are illustrated. Note that the probability of failure is not determined through the overlap of the two curves. In Figure 21 B) the density function for the failure probability is illustrated as a function of the realizations of the random variables R and S. The integral under this density is not equal to 1 but equal to the failure probability P_f. Finally from Figure 21 B) the most likely combination of values of r and s leading to failures may be identified by the mode of the density function for the probability of failure. This point is also often referred to as the design point r^*, s^*.

There exists no general closed form solution to the integral in Equation (109) but for a number of special cases solutions may be derived. One case is when both the resistance variable R and the load variable S are normally distributed. In this case the failure probability may be assessed directly by considering the random variable M often referred to as the safety margin

$$M = R - S \tag{110}$$

whereby the probability of failure may be assessed through

$$P_F = P(R - S \le 0) = P(M \le 0) \tag{111}$$

where M is also being normal distributed with parameters $\mu_M = \mu_R - \mu_S$ and standard deviation

$\sigma_M = \sqrt{\sigma_R^2 + \sigma_S^2}$.

The failure probability may now be determined by use of the standard normal distribution function as

$$P_F = \Phi(\frac{0-\mu_M}{\sigma_M}) = \Phi(-\beta) \tag{112}$$

where $\mu_M / \sigma_M = \beta$ is called the safety index. The geometrical interpretation of the safety index is illustrated in Figure 22.

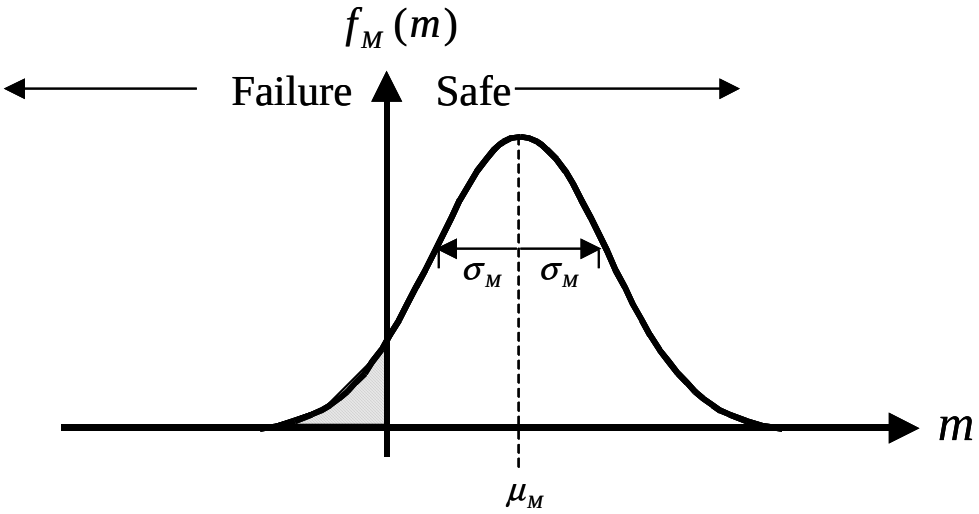


Figure 22 Illustration of the probability density function for the normally distributed safety margin M.

From Figure 22 it is seen that the safety index β may be interpreted as the number of standard deviation by which the mean value of the safety margin M exceeds zero, or equivalently the distance from the mean value of the safety margin to the most likely failure point.

As indicated previously closed form solutions may also be obtained for other special cases. However, as numerical methods have been developed for the purpose of solving Equation (109) we will not consider these in the further.

In the general case the resistance and the load cannot be described by only two random variables but rather by functions of random variables.

$$\begin{aligned} R &= f_1(\mathbf{X}) \\ S &= f_2(\mathbf{X}) \end{aligned} \tag{113}$$

where X is a vector with n so-called basic random variables. As indicated in Equation (113) both the resistance and the loading may be a function of the same random variables and R and S may thus be statistically dependent.

Furthermore the safety margin

$$M = R - S = f_1(\mathbf{X}) - f_2(\mathbf{X}) = g(\mathbf{X}) \tag{114}$$

is in general no longer normal distributed. The function $g(\mathbf{x})$ is usually denoted the limit state function, i.e. an indicator of the state of the considered component. For realizations of the basic random variables X for which $g(\mathbf{x}) \le 0$ the component is in a state of failure and otherwise for $g(\mathbf{x}) > 0$ the component is in a safe state.

Setting $g(\mathbf{x}) = 0$ defines a (n-1) dimensional hyper surface in the space spanned by the n basic random variables. This hyper surface is denoted the failure surface and thus separates all possible realizations of the basic random variables X resulting in failure, i.e. the failure domain, from the realizations resulting in a safe state, the safe domain.

Thereby the probability of failure may be determined through the following n dimensional integral

$$P_F = \int_{g(x)\leq 0} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} \tag{115}$$

where $f_{\mathbf{X}}(\mathbf{x})$ is the joint probability density function for the vector of basic random variables X and the integration is performed over the failure domain.

The solution of the integral in Equation is by no means a trivial matter except for very special cases and in most practical applications numerical approximate approaches must be pursued. Here it shall, however, be emphasized that usual numerical integration techniques are not appropriate for the solution of the integral in Equation (115) due to the fact that the numerical effort to solve it with sufficient accuracy in case of small failure probabilities becomes overwhelming and in addition to this the integration domain is not easy to represent for such algorithms.
This matter shall not be treated further in the present context but deferred to a later chapter describing some of the basics of the so-called methods of structural reliability.

6 First Order Reliability Methods

6.1 Introduction

The first developments of First Order Reliability Methods, also known as FORM methods took place almost 30 years ago. Since then the methods have been refined and extended significantly and by now they form one of the most important methods for reliability evaluations in structural reliability theory. Several commercial computer codes have been developed for FORM analysis and the methods are widely used in practical engineering problems and for code calibration purposes.
In the present chapter first the basic idea behind the FORM methods will be highlighted and thereafter the individual steps of the methods will be explained in detail.

6.2 Failure Events and Basic Random Variables

In reliability analysis of technical systems and components the main problem is to evaluate the probability of failure corresponding to a specified reference period. However, also other non-failure states of the considered component or system may be of interest, such as excessive damage, unavailability, etc.
In general any state, which may be associated with consequences in terms of costs, loss of lives and impact to the environment are of interest. In the following we will not different between these different types of states but for simplicity refer to all these as being failure events, however, bearing in mind that also non-failure states may be considered in the same manner.

It is convenient to describe failure events in terms of functional relations, which if they are fulfilled define that the considered event will occur. A failure event may be described by a functional relation, the limit state function $g(\mathbf{x})$ in the following way

$$\mathbf{F} = \{g(\mathbf{x}) \leq 0\} \tag{116}$$

where the components of the vector $\mathbf{x}$ are realizations of the so-called basic random variables $\mathbf{X}$ representing all the relevant uncertainties influencing the probability of failure. In Equation (116) the failure event $\mathbf{F}$ is simply defined as the set of realization of the function $g(\mathbf{x})$, which are zero or negative.

As already mentioned other events than failure may be of interest in reliability analysis and e.g. in reliability updating problems also events of the following form are highly relevant

$$\mathbf{I} = \{h(\mathbf{x}) = 0\} \tag{117}$$

Having defined the failure event the probability of failure may be determined by the following integral

$$P_f = \int_{g(\mathbf{x})\leq 0} f_{\mathbf{X}}(\mathbf{x})d\mathbf{x} \tag{118}$$

where $f_{\mathbf{X}}(\mathbf{x})$ is the joint probability density function of the random variables $\mathbf{X}$. This integral is, however, non-trivial to solve and numerical approximations are expedient. Various methods for the solution of the integral in Equation (118) have been proposed including numerical integration techniques, Monte Carlo simulation and asymptotic Laplace expansions. Numerical integration techniques very rapidly become in efficient for increasing dimension of the vector $\mathbf{X}$ and are in general irrelevant. Monte Carlo simulation techniques may be efficient but in the following we will direct the focus on the widely applied and quite efficient FORM methods, which furthermore can be shown to be consistent with the solutions obtained by asymptotic Laplace integral expansions.

6.3 Linear Limit State Functions and Normal Distributed Variables

For illustrative purposes we will first consider the case where the limit state function $g(\mathbf{x})$ is a linear function of the basic random variables $\mathbf{X}$. Then we may write the limit state function as

$$g(x) = a_0 + \sum_{i=1}^{n} a_i \cdot x_i \tag{119}$$

If the basic random variables are normally distributed we furthermore have that the linear safety margin M defined through

$$M = a_0 + \sum_{i=1}^{n} a_i \cdot X_i \tag{120}$$

is also normally distributed with mean value and variance

$$\begin{aligned} \mu_M &= a_0 + \sum_{i=1}^{n} a_i \mu_{X_i} \\ \sigma_M^2 &= \sum_{i=1}^{n} a_i^2 \sigma_{X_i}^2 + \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \rho_{ij} a_i a_j \sigma_i \sigma_j \end{aligned} \tag{121}$$

where ρ_{ij} are the correlation coefficients between the variables X_i and X_j.
Defining the failure event by Equation (116) we can write the probability of failure as

$$P_F = P(g(\mathbf{X}) \leq 0) = P(M \leq 0) \tag{122}$$

which in this simple case reduces to the evaluation of the standard normal distribution function

$$P_F = \Phi(-\beta) \tag{123}$$

where β the so-called reliability index (due to Cornell [12] and Basler [3]) is given as

$$\beta = \frac{\mu_M}{\sigma_M} \tag{124}$$

The reliability index as defined in Equation (123) has a geometrical interpretation as illustrated in Figure 23 where a two dimensional case is considered.

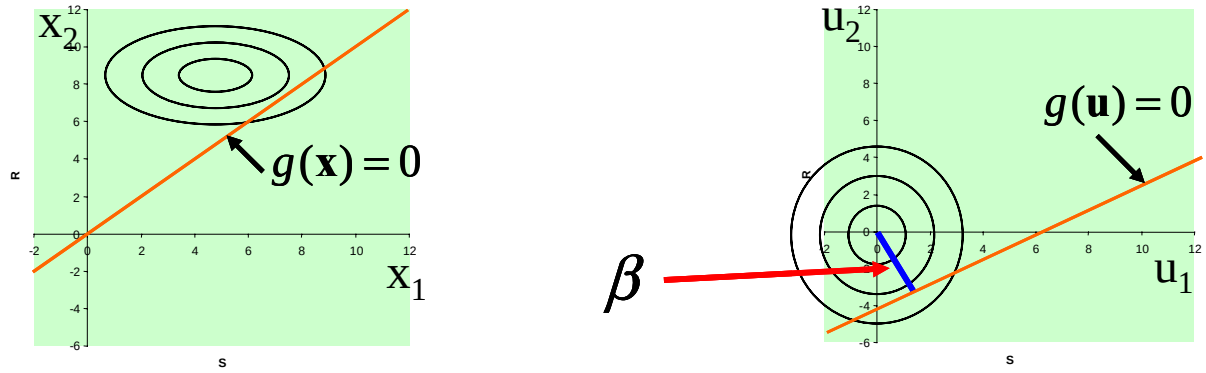


Figure 23 Illustration of the two-dimensional case of a linear limit state function and standardized normally distributed variables $\mathbf{U}$.

In Figure 23 the limit state function $g(\mathbf{x})$ has been transformed into the limit state function $g(\mathbf{u})$ by normalization of the random variables in to standardized normally distributed random variables as

$$U_i = \frac{X_i - \mu_{X_i}}{\sigma_{X_i}} \tag{125}$$

such that the random variables U_i have zero means and unit standard deviations.

Then the reliability index β has the simple geometrical interpretation as the smallest distance from the line (or generally the hyper-plane) forming the boundary between the safe domain and the failure domain, i.e. the domain defined by the failure event. It should be noted that this definition of the reliability index (due to Hasofer and Lind [6]) does not depend on the limit state function but rather the boundary between the safe domain and the failure domain. The point on the failure surface with the smallest distance to origin is commonly denoted the design point.

It is seen that the evaluation of the probability of failure in this simple case reduces to some simple evaluations in terms of mean values and standard deviations of the basic random variables, i.e. the first and second order information.

Example

Consider a steel rod under pure tension loading. The rod will fail if the applied stresses on the rod cross-sectional area exceeds the steel yield stress. The yield stress R of the rod and the loading stress on the rod S are assumed to be uncertain modeled by uncorrelated normal distributed variables. The mean values and the standard deviations of the yield strength and the loading are given as $\mu_R = 350, \sigma_R = 35$ MPa and $\mu_S = 200, \sigma_S = 40$ MPa respectively.

The limit state function describing the vent of failure may be written as

$$g(\mathbf{x}) = r - s$$

whereby the safety margin M may be written as

$$M = R - S$$

The mean value and standard deviation of the safety margin M are thus

$$\mu_M = 350 - 200 = 150$$

$$\sigma_M = \sqrt{35^2 + 40^2} = 53.15$$

whereby we may calculate the reliability index as

$$\beta = \frac{150}{53.15} = 2.84$$

Finally we have that the failure probability is determined as

$$P_F = \Phi(-2.84) = 2.4 \cdot 10^{-3}$$

6.4 Non-linear Limit State Functions

When the limit state function is not linear in the basic random variables $\mathbf{X}$ the situation is not as simple as outlined in the previous. An obvious approach is, however, to represent the failure domain in terms of a linearization of the limit state function, but the question remain how to do this appropriately.

Hasofer and Lind [7] suggested to perform this linearization in the design point of the failure surface represented in normalized space. The situation is illustrated in the two dimensional space in Figure 24.

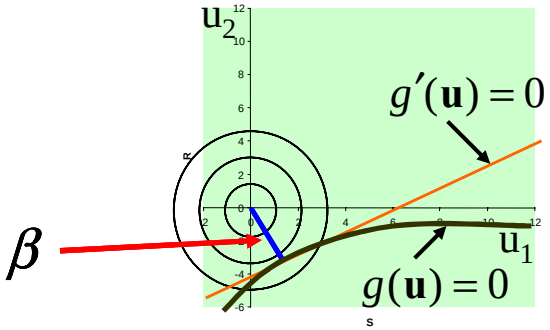


Figure 24 Illustration of the linearization proposed by Hasofer and Lind [7] in standard normal space.

In Figure 24 a principal sketch is given illustrating that the failure surface is linearized in the design point by the line $g'(\mathbf{u}) = 0$.

As the limit state function is in general non-linear one does not know the design point in advance and this has to be found iteratively e.g by solving the following optimization problem

$$\beta = \min_{\mathbf{u} \in \{g(\mathbf{u})=0\}} \sqrt{\sum_{i=1}^{n} u_i^2} \tag{126}$$

This problem may be solved in a number of different ways. Provided that the limit state function is differentiable the following simple iteration scheme may be followed

$$\alpha_i = \frac{-\frac{\partial g}{du_i}(\beta \cdot \bar{\alpha})}{\left[\sum_{j=1}^{n} \frac{\partial g}{du_i}(\beta \cdot \bar{\alpha})^2\right]^{1/2}}, \quad i = 1,2,..n \tag{127}$$

$$g(\beta \cdot \alpha_1, \beta \cdot \alpha_2, ... \beta \cdot \alpha_n) = 0$$

which in a few, say 6-10 iterations will provide the design point $\mathbf{u}^*$ as well as the reliability index β.

Example

Consider again the steel rod from the previous example. However, now it is assumed that the cross sectional areas of the steel rod A is also uncertain.

The steel yield stress R is normal distributed with mean values and standard deviation $\mu_R = 350, \sigma_R = 35$ MPa and the loading S is normal distributed with mean value and standard deviation $\mu_S = 20, \sigma_R = 4$ MPa. Finally the cross sectional area A is also assumed normally distributed with mean value and standard deviation $\mu_A = 10, \sigma_A = 2\ mm^2$.

The limit state function may be written as

$$g(\mathbf{x}) = r \cdot a - s$$

Now the first step is to transform the normally distributed random variables R, A and S into standardized normally distributed random variables, i.e.

$$U_R = \frac{R - \mu_R}{\sigma_R}$$
$$U_A = \frac{A - \mu_A}{\sigma_A}$$
$$U_S = \frac{S - \mu_S}{\sigma_S}$$

The limit state function may now be written in the space of the standardized normally distributed random variables as

$$g(u) = (u_R\sigma_R + \mu_R)(u_A\sigma_A + \mu_A) - (u_S\sigma_S + \mu_S)$$

$$= (35u_R + 350)(2u_A + 10) - (40u_S + 200)$$
$$= 350\mathrm{u}_\mathrm{R} + 700u_A - 40u_S + 70u_Ru_A + 3300$$

The reliability index and the design point may now be determined in accordance with Equation (126) as

$$\beta = \frac{-3300}{350\alpha_R + 700\alpha_A - 40\alpha_S + 70\beta\alpha_R\alpha_A}$$
$$\alpha_R = -\frac{1}{k}(350 + 70\beta\alpha_A)$$
$$\alpha_A = -\frac{1}{k}(700 + 70\beta\alpha_R)$$
$$\alpha_S = \frac{40}{k}$$

with

$$k = \sqrt{\alpha_R^2 + \alpha_A^2 + \alpha_S^2}$$

which by calculation gives the iteration history shown in Table 1

Iteration	Start	1	2	3	4	15
β	3.0000	5.4218	4.3607	4.7149	4.5973	4.6400
α_R	-0.5800	-0.3664	0.0041	-0.0647	-0.0312	-0.0382
α_A	-0.5800	-0.9283	-0.9973	-0.9961	-0.9975	-0.9963
α_S	0.5800	0.0642	0.0729	0.0597	0.0632	0.0768

Table 11 Iteration history for the non-linear limit state example

6.5 Correlated and Dependent Random Variables

The situation where basic random variables $\mathbf{X}$ are stochastically dependent is often encountered in practical problems. For normally distributed random variables we remember that the joint probability distribution function may be described in terms of the first two moments, i.e. the mean value vector and the covariance matrix. This is, however, only the case for normally or log-normally distributed random variables.
Considering in the following the case of normally distributed random variables these situations may be treated completely along the same lines as described in the foregoing. However, provided that we in addition to the transformation by which we go from a limit state function expressed in $\mathbf{X}$ variables to a limit state function expressed in $\mathbf{U}$ variables introduce an additional transformation in between where we obtain the considered random variables first are made uncorrelated before they are normalized. I.e. the row of transformations yields

$$\mathbf{X} \rightarrow \mathbf{Y} \rightarrow \mathbf{U}$$

In the following we will see how this transformation may be implemented in the iterative outlined previously.

Let us assume that the basic random variables $\mathbf{X}$ are correlated with covariance matrix given as

$$\mathbf{C}_{\mathbf{X}} = \begin{bmatrix} Var[X_1] & Cov[X_1, X_2] \ldots & Cov[X_1, X_n] \\ \vdots & \vdots & \vdots \\ Cov[X_n, X_1] & \cdots & Var[X_n] \end{bmatrix} \tag{128}$$

If only the diagonal elements of this matrix are non-zero clearly the basic random variables are uncorrelated. However, according to the theorems of linear algebra it is possible to transform the coordinate system of the $\mathbf{X}$ variables into a coordinate system spanned by $\mathbf{Y}$ variables such that the covariance matrix for the $\mathbf{Y}$ variables has the following form

$$\mathbf{C}_{\mathbf{Y}} = \begin{bmatrix} Var[Y_1] & 0 \ldots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \cdots & Var[Y_n] \end{bmatrix} \tag{129}$$

Such a transformation may be performed as

$$\mathbf{Y} = \mathbf{A}^T \mathbf{X} \tag{130}$$

where $\mathbf{A}$ is an orthogonal matrix with column vectors equal to the orthonormal eigenvectors of $\mathbf{C}_{\mathbf{X}}$.

By this transformation we achieve that the mean value vector of $\mathbf{Y}$ are given as

$$\mathbf{E}[\mathbf{Y}] = \mathbf{A}^T \mathbf{X}$$

and the variances of $\mathbf{Y}$ are equal to the eigenvalues of the matrix $\mathbf{C}_\mathbf{X}$, i.e.

$$\mathbf{C}_\mathbf{Y} = \mathbf{A}^T \mathbf{C}_\mathbf{X} \mathbf{A}$$

In case the stochastically dependent basic random variables are not normally or log-normally distributed the above described transformation is not appropriate and other transformations may be applied. The interested reader is referred to literature for further information on e.g. the Nataf transformations see e.g. Madsen et al. [4].

In the following we will not consider the numerical implications of the transformation described in the foregoing as these become somewhat involving for hand calculations. Standard commercial software for FORM analysis include these transformations as an option.

6.6 Non-Normal Distributed Random Variables

As a further development of the iterative calculation scheme for the evaluation of the failure probability we need to consider the cases where also non-normally distributed random variables are present.

One of the commonly used approaches for treating this situation is to approximate the probability distribution function and the probability density function for the non-normally distributed random variables by normal distribution and normal density functions.
As the design point is usually located in the tails of the distribution functions of the basic random variables the scheme is often referred to as the "normal tail approximation".

Denoting by $\mathbf{x}^*$ the design point the approximation is introduced by

$$F_{X_{ii}}(x_i^*) = \Phi\left(\frac{x_i^* - \mu'_{X_i}}{\sigma'_{X_i}}\right) \tag{131}$$

$$f_{X_{ii}}(x_i^*) = \frac{1}{\sigma_{X_i}} \varphi\left(\frac{x_i^* - \mu'_{X_i}}{\sigma'_{X_i}}\right) \tag{132}$$

where μ'_{X_i} and σ_{X_i} are the unknown mean value and standard deviation of the approximating normal distribution.

Solving Equation (131) and (132) with respect to μ'_{X_i} and σ_{X_i} we obtain

$$\sigma'_{X_i} = \frac{\varphi(\Phi^{-1}(F_{X_i}(x_i^*)))}{f_{X_i}(x_i^*)}$$

$$\mu'_{X_i} = x_i^* - \Phi^{-1}(F_{X_i}(x_i^*))\sigma'_{X_i} \tag{133}$$

This transformation may be introduced in the iterative evaluation of the reliability index as a final step before the basic random variables are normalized.

6.7 Software for FORM analysis

Several software packages are available for the FORM analysis following the principles out lined in the forgoing sections. Most of the programs are more or less self-explanatory provided that the basic principles of FORM analysis are known.

The reader is referred to software packages such as STRUREL and VAP for which more information is available on the www.

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