# Methods of Structural Reliability Theory – an Introduction

## Nachdiplomkurs Risiko und Sicherheit

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# First Order Reliability Methods

## 1. Introduction

The first developments of First Order Reliability Methods, also known as FORM methods took place almost 30 years ago. Since then the methods have been refined and extended significantly and by now they form one of the most important methods for reliability evaluations in structural reliability theory. Several commercial computer codes have been developed for FORM analysis and the methods are widely used in practical engineering problems and for code calibration purposes.
In the present chapter first the basic idea behind the FORM methods will be highlighted and thereafter the individual steps of the methods will be explained in detail.

## 2. Failure Events and Basic Random Variables

In reliability analysis of technical systems and components the main problem is to evaluate the probability of failure corresponding to a specified reference period. However, also other non-failure states of the considered component or system may be of interest, such as excessive damage, unavailability, etc.
In general any state, which may be associated with consequences in terms of costs, loss of lives and impact to the environment are of interest. In the following we will not different between these different types of states but for simplicity refer to all these as being failure events, however, bearing in mind that also non-failure states may be considered in the same manner.

It is convenient to describe failure events in terms of functional relations, which if they are fulfilled define that the considered event will occur. A failure event may be described by a functional relation, the limit state function $g(\mathbf{x})$ in the following way

$$\mathbf{F} = \{g(\mathbf{x}) \leq 0\} \tag{1}$$

where the components of the vector $\mathbf{x}$ are realisations of the so-called basic random variables $\mathbf{X}$ representing all the relevant uncertainties influencing the probability of failure. In Equation (1) the failure event $\mathbf{F}$ is simply defined as the set of realisation of the function $g(\mathbf{x})$, which are zero or negative.

As already mentioned other events than failure may be of interest in reliability analysis and e.g. in reliability updating problems also events of the following form are highly relevant

$$\mathbf{I} = \{h(\mathbf{x}) = 0\} \tag{2}$$

Having defined the failure event the probability of failure may be determined by the following integral

$$P_f = \int_{g(\mathbf{x})\leq 0} f_{\mathbf{X}}(\mathbf{x})d\mathbf{x} \tag{3}$$

where $f_{\mathbf{X}}(\mathbf{x})$ is the joint probability density function of the random variables $\mathbf{X}$. This integral is, however, non-trivial to solve and numerical approximations are expedient. Various methods for the solution of the integral in Equation (3) have been proposed including numerical integration techniques, Monte Carlo simulation and asymptotic Laplace expansions. Numerical integration techniques very rapidly become in efficient for increasing dimension of the vector $\mathbf{X}$ and are in general irrelevant. Monte Carlo simulation techniques may be efficient but in the following we will direct the focus on the widely applied and quite efficient FORM methods, which furthermore can be shown to be consistent with the solutions obtained by asymptotic Laplace integral expansions.

## 3. Linear Limit State Functions and Normal Distributed Variables

For illustrative purposes we will first consider the case where the limit state function $g(\mathbf{x})$ is a linear function of the basic random variables $\mathbf{X}$. Then we may write the limit state function as

$$g(x) = a_0 + \sum_{i=1}^{n} a_i \cdot x_i \tag{4}$$

If the basic random variables are normally distributed we furthermore have that the linear safety margin $M$ defined through

$$M = a_0 + \sum_{i=1}^{n} a_i \cdot X_i \tag{5}$$

is also normally distributed with mean value and variance

$$\begin{aligned} \mu_M &= a_0 + \sum_{i=1}^{n} a_i \mu_{X_i} \\ \sigma_M^2 &= \sum_{i=1}^{n} a_i^2 \sigma_{X_i}^2 + \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \rho_{ij} a_i a_j \sigma_i \sigma_j \end{aligned} \tag{6}$$

where $\rho_{ij}$ are the correlation coefficients between the variables $X_i$ and $X_j$.
Defining the failure event by Equation (1) we can write the probability of failure as

$$P_F = P(g(\mathbf{X}) \le 0) = P(M \le 0) \tag{7}$$

which in this simple case reduces to the evaluation of the standard normal distribution function

$$P_F = \Phi(-\beta) \tag{8}$$

where $\beta$ the so-called reliability index (due to Cornell [1] and Basler [2] ) is given as

$$\beta = \frac{\mu_M}{\sigma_M} \tag{9}$$

The reliability index as defined in Equation (8) has a geometrical interpretation as illustrated in Figure 1 where a two dimensional case is considered.

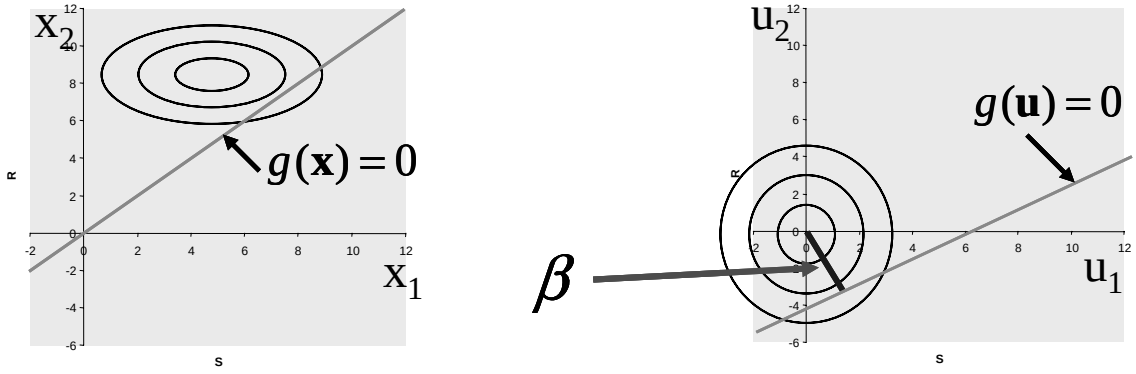


Figure 1 Illustration of the two-dimensional case of a linear limit state function and standardised normally distributed variables $\mathbf{U}$ .

In Figure 1 the limit state function $g(\mathbf{x})$ has been transformed into the limit state function $g(\mathbf{u})$ by normalisation of the random variables in to standardized normally distributed random variables as

$$U_i = \frac{X_i - \mu_{X_i}}{\sigma_{X_i}} \tag{10}$$

such that the random variables $U_i$ have zero means and unit standard deviations.

Then the reliability index $\beta$ has the simple geometrical interpretation as the smallest distance from the line (or generally the hyper-plane) forming the boundary between the safe domain and the failure domain, i.e. the domain defined by the failure event. It should be noted that this definition of the reliability index (due to Hasofer and Lind [3] does not depend on the limit state function but rather the boundary between the safe domain and the failure domain. The point on the failure surface with the smallest distance to origin is commonly denoted the design point.

It is seen that the evaluation of the probability of failure in this simple case reduces to some simple evaluations in terms of mean values and standard deviations of the basic random variables, i.e. the first and second order information.

Example
Consider a steel rod under pure tension loading. The rod will fail if the applied stresses on the rod cross-sectional area exceeds the steel yield stress. The yield stress $R$ of the rod and the loading stress on the rod $S$ are assumed to be uncertain modelled by uncorrelated normal distributed variables. The mean values and the standard deviations of the yield strength and the loading are given as $\mu_R = 350, \sigma_R = 35$ MPa and $\mu_S = 200, \sigma_S = 40$ MPa respectively.

The limit state function describing the vent of failure may be written as

$$g(\mathbf{x}) = r - s$$

whereby the safety margin $M$ may be written as

$$M = R - S$$

The mean value and standard deviation of the safety margin M are thus

$$\mu_M = 350 - 200 = 150$$

$$\sigma_M = \sqrt{35^2 + 40^2} = 53.15$$

whereby we may calculate the reliability index as

$$\beta = \frac{150}{53.15} = 2.84$$

Finally we have that the failure probability is determined as

$P_F = \Phi(-2.84) = 2.4 \cdot 10^{-3}$

## 4. Non-linear Limit State Functions

When the limit state function is not linear in the basic random variables $\mathbf{X}$ the situation is not as simple as outlined in the previous. An obvious approach is, however, to represent the failure domain in terms of a linearization of the limit state function, but the question remain how to do this appropriately.

Hasofer and Lind [3] suggested to perform this linearization in the design point of the failure surface represented in normalised space. The situation is illustrated in the two dimensional space in Figure 2.

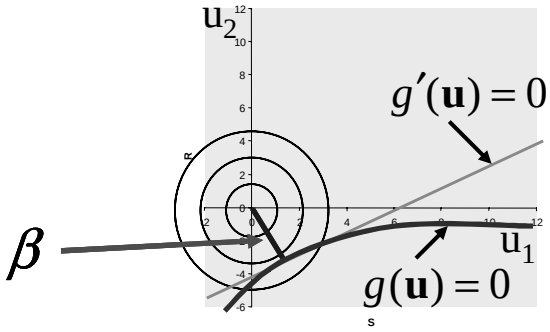


Figure 2 Illustration of the linearization proposed by Hasofer and Lind [3] in standard normal space.

In Figure 2 a principal sketch is given illustrating that the failure surface is linearized in the design point by the line $g'(\mathbf{u}) = 0$.

As the limit state function is in general non-linear one does not know the design point in advance and this has to be found iteratively e.g by solving the following optimisation problem

$$\beta = \min_{\mathbf{u} \in \{g(\mathbf{u})=0\}} \sqrt{\sum_{i=1}^{n} u_i^2} \tag{11}$$

This problem may be solved in a number of different ways. Provided that the limit state function is differentiable the following simple iteration scheme may be followed

$$\alpha_i = \frac{-\dfrac{\partial g}{du_i}(\beta \cdot \bar{\alpha})}{\left[\sum_{j=1}^{n} \dfrac{\partial g}{du_i}(\beta \cdot \bar{\alpha})^2\right]^{1/2}}, \quad i = 1,2,..n \qquad (12)$$

$$g(\beta \cdot \alpha_1, \beta \cdot \alpha_2, ... \beta \cdot \alpha_n) = 0$$

which in a few, say 6-10 iterations will provide the design point $\mathbf{u}^*$ as well as the reliability index $\beta$.

Example
Consider again the steel rod from the previous example. However, now it is assumed that the cross sectional areas of the steel rod $A$ is also uncertain.
The steel yield stress $R$ is normal distributed with mean values and standard deviation $\mu_R = 350, \sigma_R = 35$ MPa and the loading $S$ is normal distributed with mean value and standard deviation $\mu_S = 20, \sigma_R = 4$ MPa. Finally the cross sectional area $A$ is also assumed normally distributed with mean value and standard deviation $\mu_A = 10, \sigma_A = 2\ mm^2$.

The limit state function may be written as

$$g(\mathbf{x}) = r \cdot a - s$$

Now the first step is to transform the normally distributed random variables $R$, $A$ and $S$ into standardized normally distributed random variables, i.e.

$$U_R = \frac{R - \mu_R}{\sigma_R}$$
$$U_A = \frac{A - \mu_A}{\sigma_A}$$
$$U_S = \frac{S - \mu_S}{\sigma_S}$$

The limit state function may now be written in the space of the standardized normally distributed random variables as

$$g(u) = (u_R \sigma_R + \mu_R)(u_A \sigma_A + \mu_A) - (u_S \sigma_S + \mu_S)$$

$$= (35u_R + 350)(2u_A + 10) - (40u_S + 200)$$
$$= 350\mathrm{u}_\mathrm{R} + 700u_A - 40u_S + 70u_R u_A + 3300$$

The reliability index and the design point may now be determined in accordance with Equation (11) as

$$\beta = \frac{-3300}{350\alpha_R + 700\alpha_A - 40\alpha_S + 70\beta\alpha_R\alpha_A}$$

$$\alpha_R = -\frac{1}{k}(350 + 70\beta\alpha_A)$$

$$\alpha_A = -\frac{1}{k}(700 + 70\beta\alpha_R)$$

$$\alpha_S = \frac{40}{k}$$

with

$$k = \sqrt{\alpha_R^2 + \alpha_A^2 + \alpha_S^2}$$

which by calculation gives the iteration history shown in Table 1

| Iteration | Start | 1 | 2 | 3 | 4 | 15 |
|---|---|---|---|---|---|---|
| β | 3.0000 | 5.4218 | 4.3607 | 4.7149 | 4.5973 | 4.6400 |
| $\alpha_R$ | -0.5800 | -0.3664 | 0.0041 | -0.0647 | -0.0312 | -0.0382 |
| $\alpha_A$ | -0.5800 | -0.9283 | -0.9973 | -0.9961 | -0.9975 | -0.9963 |
| $\alpha_S$ | 0.5800 | 0.0642 | 0.0729 | 0.0597 | 0.0632 | 0.0768 |

Table 1 Iteration history for the non-linear limit state example

## 5. Correlated and Dependent Random Variables

The situation where basic random variables $\mathbf{X}$ are stochastically dependent is often encountered in practical problems. For normally distributed random variables we remember that the joint probability distribution function may be described in terms of the first two moments, i.e. the mean value vector and the covariance matrix. This is, however, only the case for normally or log-normally distributed random variables.

Considering in the following the case of normally distributed random variables these situations may be treated completely along the same lines as described in the foregoing. However, provided that we in addition to the transformation by which we go from a limit state function expressed in $\mathbf{X}$ variables to a limit state function expressed in $\mathbf{U}$ variables introduce an additional transformation in between where we obtain the considered random variables first are made uncorrelated before they are normalised. I.e. the row of transformations yields

$$\mathbf{X} \rightarrow \mathbf{Y} \rightarrow \mathbf{U}$$

In the following we will see how this transformation may be implemented in the iterative outlined previously.

Let us assume that the basic random variables $\mathbf{X}$ are correlated with covariance matrix given as

$$\mathbf{C}_{\mathbf{X}} = \begin{bmatrix} Var[X_1] & Cov[X_1, X_2] \ldots & Cov[X_1, X_n] \\ \vdots & \vdots & \vdots \\ Cov[X_n, X_1] & \cdots & Var[X_n] \end{bmatrix} \tag{13}$$

If only the diagonal elements of this matrix are non-zero clearly the basic random variables are uncorrelated. However, according to the theorems of linear algebra it is possible to transform the coordinate system of the $\mathbf{X}$ variables into a coordinate system spanned by $\mathbf{Y}$ variables such that the covariance matrix for the $\mathbf{Y}$ variables has the following form

$$\mathbf{C}_{\mathbf{Y}} = \begin{bmatrix} Var[Y_1] & 0 \ldots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \cdots & Var[Y_n] \end{bmatrix} \tag{14}$$

Such a transformation may be performed as

$$\mathbf{Y} = \mathbf{A}^T \mathbf{X} \tag{15}$$

where $\mathbf{A}$ is an orthogonal matrix with column vectors equal to the orthonormal eigenvectors of $\mathbf{C}_{\mathbf{X}}$.

By this transformation we achieve that the mean value vector of $\mathbf{Y}$ are given as

$$\mathbf{E}[\mathbf{Y}] = \mathbf{A}^T \mathbf{X}$$

and the variances of $\mathbf{Y}$ are equal to the eigenvalues of the matrix $\mathbf{C}_{\mathbf{X}}$, i.e.

$$\mathbf{C}_{\mathbf{Y}} = \mathbf{A}^T \mathbf{C}_{\mathbf{X}} \mathbf{A}$$

In case the stochastically dependent basic random variables are not normally or log-normally distributed the above described transformation is not appropriate and other transformations may be applied. The interested reader is referred to literature for further information on e.g. the Nataf transformations see e.g. Madsen et al. [4].

In the following we will not consider the numerical implications of the transformation described in the foregoing as these become somewhat involving for hand calculations. Standard commercial software for FORM analysis include these transformations as an option.

## 6. Non-Normal Distributed Random Variables

As a further development of the iterative calculation scheme for the evaluation of the failure probability we need to consider the cases where also non-normally distributed random variables are present.

One of the commonly used approaches for treating this situation is to approximate the probability distribution function and the probability density function for the non-normally distributed random variables by normal distribution and normal density functions.
As the design point is usually located in the tails of the distribution functions of the basic random variables the scheme is often referred to as the “normal tail approximation”.

Denoting by $\mathbf{x}^*$ the design point the approximation is introduced by

$$F_{X_{ii}}(x_i^*) = \Phi\left(\frac{x_i^* - \mu'_{X_i}}{\sigma'_{X_i}}\right) \tag{16}$$

$$f_{X_{ii}}(x_i^*) = \frac{1}{\sigma_{X_i}}\varphi\left(\frac{x_i^* - \mu'_{X_i}}{\sigma'_{X_i}}\right) \tag{17}$$

where $\mu'_{X_i}$ and $\sigma_{X_i}$ are the unknown mean value and standard deviation of the approximating normal distribution.

Solving Equation (16) and (17) with respect to $\mu'_{X_i}$ and $\sigma_{X_i}$ we obtain

$$\sigma'_{X_i} = \frac{\varphi(\Phi^{-1}(F_{X_i}(x_i^*)))}{f_{X_i}(x_i^*)}$$

$$\mu'_{X_i} = x_i^* - \Phi^{-1}(F_{X_i}(x_i^*))\sigma'_{X_i} \tag{18}$$

This transformation may be introduced in the iterative evaluation of the reliability index as a final step before the basic random variables are normalised.

## 7. Software for FORM analysis

Several software packages are available for the FORM analysis following the principles out lined in the forgoing sections. Most of the programs are more or less self-explanatory provided that the basic principles of FORM analysis are known.

The reader is referred to software packages such as STRUREL and VAP for which more information is available on the www.

## 8 .References

[1] Cornell, C. A. A Probability Based Structural Code, ACI-Journal, Vol. 66, 1969, pp. 974-985.

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[4] Madsen, H.O., Krenk, S. and Lind, N.C., Methods of Structural Safety, Prentice Hal, Inc. Englewood Cliffs, NJ 07632, 1986.